Department of Computer Science,
University of Otago

Technical Report OUCS-2010-03

Young classes of permutations

Author:

Michael Albert

Department of Computer Science, University of Otago, New Zealand

Status:

Submitted to the Journal of Algebraic Combinatorics

(also should appear in the next release on the arXiv.)

Department of Computer Science,
University of Otago, PO Box 56, Dunedin, Otago, New Zealand

http://www.cs.otago.ac.nz/research/techreports.php

YOUNG CLASSES OF PERMUTATIONS

MICHAEL ALBERT

Abstract. We characterise those classes of permutations having the property that for every tableau shape either every permutation of that shape or no permutation of that shape belongs to the class. The characterisation is in terms of the dominance order for partitions (and their conjugates) and shows that for any such class there is a constant k such that no permutation in the class can contain both an increasing and a decreasing sequence of length k.

1. Introduction

The most well known notion of the *shape* of a permutation arises from the Robinson-Schensted-Knuth correspondence, whereby the shape of $\pi \in \mathcal{S}_n$ is taken to be the shape of either of the Young tableaux produced from it by the RSK-algorithm. On the other hand, we might think of the shape of a permutation as being described by its graph.

In the first context, subshapes can be defined in terms of various natural orderings on the set of partitions. In the second context, if a permutation $\pi \in \mathcal{S}_n$ (thought of as its sequence of values $\pi_1\pi_2 \cdots \pi_n$) contains a subsequence order-isomorphic to a permutation σ then we say that σ *occurs as a pattern in*, or, *is involved in* π.

It seems natural to investigate the manner in which these two concepts interact. Unfortunately, the basic answer is: not very well. In particular Adin and Roichman [1] began such an investigation and reported indicative counterexamples to some tempting false conjectures, as well as more positive results relating to rectangular and hook shapes. More recently, at the Permutation Patterns 2010 conference at Dartmouth College, Panova [3] and Tiefenbruck [7] reported on results in this area when attention is restricted to permutations avoiding certain specific patterns.

A downward-closed (with respect to involvement) set of permutations is called a *permutation class* (or simply *class*). In this paper we investigate the question:

> Which classes, $\mathcal{C}$, of permutations have the property that if π and π' are two permutations of the same shape then either both, or neither belong to $\mathcal{C}$?

Such classes are precisely those closed in the natural equivalence relation on permutations of "having the same shape". Classes of this type will be called a *Young classes*. The most notable consequence of our characterisation of these classes is that for any proper Young class there is a constant k such that no permutation in the class contains both increasing and decreasing subsequences of length k.

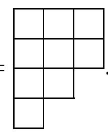
1991 *Mathematics Subject Classification.* 05A05, 05A17, 05E99.
Key words and phrases. permutation, partition, Robinson-Schensted-Knuth algorithm.

2. Definitions

We refer the reader to either [4] or [6] for more detailed discussion of the RSK-correspondence and related matters. In this section we will collect only the basic definitions and results needed for the remainder of this paper.

A *partition*, λ, is a weakly decreasing sequence of positive integers, i.e. $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. We say that λ is a partition of n, and write $|\lambda| = n$ where $n = \sum_i \lambda_i$. The individual values λ_i are called the *parts* of λ. When writing a partition as a sequence, repeated values may be represented by exponentiation. A partition is often represented by its *Young diagram*, an array of boxes of the appropriate lengths (we use the English or "top down" convention.) For instance:

$$(3,3,2,1) = (3^2,2,1) =$$



.

It is sometimes convenient to pretend that a partition may be extended by (any number of) trailing 0's. The *conjugate partition*, λ^* of a partition λ is the sequence given by:

$$\lambda^*_i = |\{j \,:\, \lambda_j \geq i\}| \,.$$

The *sum* $\lambda + \mu$ of two partitions is defined in the obvious way (its i^{th} element is $\lambda_i + \mu_i$), and their *conjugate sum*

$$\lambda +^* \mu = (\lambda^* + \mu^*)^* \,.$$

On Young diagrams these operations correspond to concatenating the two partitions horizontally or vertically, and then shoving blocks leftwards or upwards to eliminate gaps.

If λ and μ are partitions, we say that μ *dominates* λ and write $\lambda \trianglelefteq \mu$ if, for all $k \geq 1$:

$$\sum_{i=1}^{k} \lambda_k \leq \sum_{i=1}^{k} \mu_k.$$

If both $\lambda \trianglelefteq \mu$ and $\lambda^* \trianglelefteq \mu^*$ then we say that μ *doubly dominates* λ and write $\lambda \preceq \mu$. As is well known, if $|\lambda| = |\mu|$ and λ doubly dominates μ then $\lambda = \mu$.

The *shape* of permutation π (of length n) is the partition λ of n obtained when constructing its Young tableaux via the RSK-correspondence. In this case we write $\mathrm{sh}(\pi) = \lambda$.

A permutation σ *is involved in* π if the terms of some subsequence of the values of π (of the same length as σ) occur in the same relative order as the sequence of values of σ. In this case we write $\sigma \preceq \pi$. For example, $231 \preceq 421635$ because of the subsequence 463. A *permutation class* is a set of permutations closed downwards under $\preceq$. A permutation class can also be described as the set of permutations that do not involve (or *avoid*) some given set of permutations X. In that case we write $\mathcal{C} = \mathrm{Av}(X)$. If X is an antichain with respect to $\preceq$ then it is called the *basis* of the class $\mathcal{C}$.

A *Young class* is a class of permutations such that for each partition λ either all the permutations of shape λ or none of them belong to the class. The prototypical examples of Young classes are the collection of all permutations not including an increasing subsequence of some fixed length a – by Schensted's theorem [5], the shapes of all such permutations have every part smaller than a, and the converse also holds.

The use of Greek letters for both partitions and permutations is unfortunately standard. We will attempt to minimise the resulting confusion by reserving λ and μ for partitions.

3. Greene's Theorem and double domination

Greene's generalisation [2] (or see Theorem 3.5.3 of [4]) of Schensted's theorem provides a link between the shape of a permutation π and the involvement of certain patterns in π.

Theorem 1. *For any permutation π of tableau shape λ, and any k, the longest subpermutation of π that can be written as a union of k increasing subsequences has length $\lambda_1+\lambda_2+\cdots+\lambda_k$, and the longest subpermutation of π that can be written as a union of k decreasing subsequences has length $\lambda_1^*+\lambda_2^*+\cdots+\lambda_k^*$.*

To make this link more explicit, note that the permutations that avoid $(k+1)k\cdots 321$ are precisely those which can be written as a union of k increasing subsequences. So $\lambda_1+\lambda_2+\cdots+\lambda_k$ is the maximum length of an element of $\mathrm{Av}((k+1)k\cdots 321)$ which is involved in π. Of course a dual statement applies with reference to the second part of the theorem and the class $\mathrm{Av}(123\cdots(k+1))$.

The following corollary to Greene's theorem is immediate:

Corollary 2. *Let σ and π be permutations. If $\sigma \preceq \pi$ then* $\mathrm{sh}(\sigma) \preceq \mathrm{sh}(\pi)$.

Proof. This is simply because, if $\sigma \preceq \pi$ then, for any k the longest subsequence of π which can be written as the union of k increasing (decreasing) subsequences is at least as long as the corresponding subsequence of σ, since that subsequence occurs in π. □

A strong converse to the corollary is easily seen to be false. For example $(2,2) \preceq (3,1,1)$ but the permutation 54123 of shape $(3,1,1)$ has no subpermutation of shape $(2,2)$. However, even the weaker converse "if $\lambda \preceq \mu$ then for some permutations σ and π with $\mathrm{sh}(\sigma)=\lambda$ and $\mathrm{sh}(\pi)=\mu$, $\sigma \preceq \pi$" is also false. For, take $\lambda=(2,2,2)$ and $\mu=(4,1,1,1,1)$. If σ has shape λ, then to extend it to a permutation having a longest increasing subsequence of length 4 a pair of elements in increasing order must be added. However, to extend it to a permutation whose longest decreasing subsequence has length 5 a pair of elements in decreasing order must be added. To accomplish both therefore requires adding at least three elements, and thus rules out having shape μ. On the other hand, we will see below that if $\lambda \preceq \mu$ and $|\mu| = |\lambda|+1$ then there are such permutations σ and π. This will be the central result that leads to a characterisation of Young classes.

The double domination relation on partitions has obvious covering relationships (when $\lambda \preceq \mu$ and $|\mu|=|\lambda|+1$). These covering relations are of two types. Either λ is obtained from μ simply by decreasing the size of one part by one or for some $i<j<k$, λ and μ agree except that $\lambda_i=\mu_i-1$, $\lambda_j=\mu_j+1$ and $\lambda_k=\mu_k-1$.

For our subsequent arguments we need to show that these are the only covers for the relation $\preceq$ on permutations. That is:

Proposition 3. *Suppose that $\lambda \preceq \mu$, and $|\mu| - |\lambda| > 1$. Then there exists μ' with $\lambda \preceq \mu' \preceq \mu$ and $|\mu| - |\mu'| = 1$.*

Proof. We begin by eliminating some trivial or irrelevant cases. If we can obtain $\mu' \succeq \lambda$ by simply deleting a cell from μ then the result is trivial. If $\lambda_1 = \mu_1$ or $\lambda_1^* = \mu_1^*$ then the result will follow inductively by ignoring the first part (or the first part of the conjugate) in λ and μ. So, we may assume that $\mu_1 > \lambda_1$, $\mu_1^* > \lambda_1^*$, and for some $r > 1$, $\sum_{i=1}^r \mu_i = \sum_{i=1}^r \lambda_i$ (otherwise we could delete an element in the rightmost column of μ and still doubly dominate λ). Choose the least such r. Certainly $\mu_r < \lambda_r$. Similarly, we may assume that for some $c > 1$, $\sum_{i=1}^c \mu_i^* = \sum_{i=1}^c \lambda_i^*$. Again we will choose the least such c and observe that $\mu_c^* < \lambda_c^*$.

Suppose that $\mu_r \geq c$ and $\mu_c^* \geq r$, i.e. that μ contains the upper left $r \times c$ rectangle. Since $\lambda_r > \mu_r$ and $\lambda_c > \mu_c$, λ also contains the upper left $r \times c$ rectangle. Furthermore, the total number of cells of λ lying in or above row r and in or left of column c equals that of μ (since the sum of the first r rows and the first c columns agree and the overlap is fully occupied). So, the part of μ strictly below and to the right of that rectangle doubly dominates the corresponding part of λ, and we may inductively construct μ' there.

So we may now assume that either $\mu_r < c$ or $\mu_c^* < r$ and hence that there is at least one square in the upper left $r \times c$ rectangle not belonging to μ. Form μ' by removing a cell from the rightmost column, and a cell from the bottommost row of μ and adding a cell within the upper left $r \times c$ rectangle. Then certainly $\mu' \preceq \mu$ and we claim that $\lambda \preceq \mu'$. Consider first the row sums of μ'. Some among the first $r - 1$ may be one smaller than the corresponding sums of μ but, by the choice of r they are still at least as large as those of λ. From rows r through to the final row they equal those of μ, and in the final row are again one smaller. However as $|\mu| > |\lambda|$ they are all at least as great as those of λ. The argument that the column sums of μ' dominate those of λ is exactly similar. □

4. Young classes

To characterise Young classes we need to introduce two more operations on permutations, and show how they affect the shape of the resulting tableaux. Let permutations α and β of lengths n and k be given. Define:

$$\begin{aligned} \alpha \oplus \beta &= (\alpha_1, \ldots, \alpha_n, n + \beta_1, \ldots, n + \beta_k) \\ \alpha \ominus \beta &= (k + \alpha_1, \ldots, k + \alpha_n, \beta_1, \ldots, \beta_k) \end{aligned}$$

Informally, $\alpha \oplus \beta$ stacks β above and to the right of α, while $\alpha \ominus \beta$ stacks β below and to the right of α.

Proposition 4. *The operations $\oplus$ and $\ominus$ affect shape as follows:*

$$\begin{aligned} \mathrm{sh}(\alpha \oplus \beta) &= \mathrm{sh}(\alpha) + \mathrm{sh}(\beta) \\ \mathrm{sh}(\alpha \ominus \beta) &= \mathrm{sh}(\alpha) +^* \mathrm{sh}(\beta). \end{aligned}$$

Proof. The proof of both parts is most easily seen by considering Greene's theorem. In $\alpha \oplus \beta$ a subsequence that is the union of k increasing sequences can be divided into such a subsequence of α and such a subsequence of β (and the union of any two such subsequences is again of the same type). So, the sum of the first k parts of $\mathrm{sh}(\alpha \oplus \beta)$ is equal to the sum of the first k parts of $\mathrm{sh}(\alpha)$ plus the sum of the first k parts of $\mathrm{sh}(\beta)$. The result follows immediately. The corresponding result for $\ominus$ likewise follows by considering unions of decreasing subsequences.

□

We remark that the result for $\oplus$ is also easily seen by considering the actual bumping operations of the RSK algorithm, but the result for $\ominus$ is not so transparent in that context.

Before embarking on the main theorem, we prove a special case which will be used there:

Lemma 5. *Let $n \geq 2$. There exist permutations σ of shape (n, n) and π of shape $(n+1, n-1, 1)$ with $\sigma \preceq \pi$.*

Proof. Take
$$\sigma = (n+1, 1, n+2, 2, n+3, 3, \ldots, 2n, n),$$
and take
$$\pi = (n+2, 1, n+1, 2, n+3, \ldots, 2n+1, n).$$
That is π is obtained from σ by inserting a single new point immediately to the right of the original 1, and immediately below the original $n+1$ (all values from $n+1$ upwards thereby increasing by one). Obviously $\sigma \preceq \pi$ and the shape of σ is (n, n). The permutation π has a decreasing sequence $(n+2, n+1, 2)$ of length 3 so its shape has three parts. It also has an increasing sequence of length $n+1$ (starting $1, n+1, n+3$) but cannot have a longer one since it is obtained from σ by adding a single element. As π contains σ the sum of the first two parts of its shape must be at least $2n$. All together this implies the desired result. Alternatively of course we could simply have considered the RSK algorithm acting on π. □

Corollary 6. *Let $n, k \geq 2$. There exist permutations θ of shape (n^k) and τ of shape $(n+1, n^{k-2}, n-1, 1)$ with $\theta \preceq \tau$.*

Proof. Take π and σ as given by the preceding proposition. Let α be any permutation of shape (n^{k-2}). Then we can take $\theta = \pi \ominus \alpha$ and $\tau = \sigma \ominus \alpha$. □

Theorem 7. *Let λ and μ be partitions of n and $n+1$ respectively. Then there exist permutations σ and π of shapes λ and μ respectively, and with $\sigma \preceq \pi$ if and only if $\lambda \preceq \mu$.*

Proof. One direction follows immediately from Corollary 2.

For the other implication, there are two cases to consider: first λ might be contained in μ, second this might fail.

The first case is easy. Suppose that $\lambda_j = \mu_j$ for all j except $\lambda_i + 1 = \mu_i$. Take any permutation α of shape $(\lambda_1, \lambda_2, \ldots, \lambda_{i-1})$ and any permutation β of shape $(\lambda_i, \lambda_{i+1}, \ldots)$. Then $\alpha \ominus \beta$ has shape λ. On the other hand $\alpha \ominus (\beta \oplus 1)$ has shape σ and involves λ.

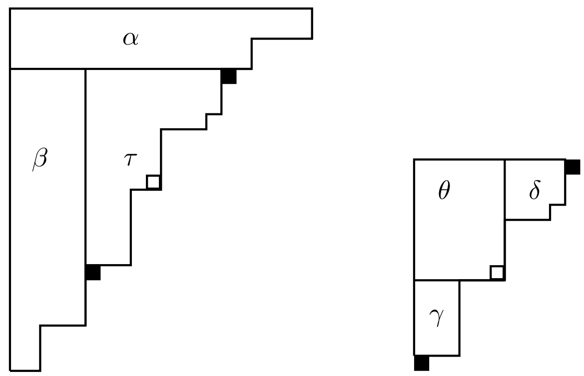


FIGURE 1. The splittings required to reduce the proof of Theorem 7 to that of the preceding corollary. The indicated white cell is to be removed and replaced by the two black ones.

For the second part we will make use of the preceding lemma. Recall that in this case there are indices $i < j < k$ with $\mu_i = \lambda_i + 1$, $\mu_j = \lambda_j - 1$ and $\mu_k = \lambda_k + 1$. Informally, the cell at the end of row j in the Young diagram of λ splits into two cells, one of which moves to a higher row, and the other to a lower row. Using $\oplus$ and $\ominus$ decompositions as above we can assume that $i = 1$, and that $\mu_k = 1$ is the last part of μ. To see that this is possible, refer to Figure 1. We will choose arbitrary permutations α and β of the indicated shapes and seek permutations τ and τ' so that:

$$\tau \preceq \tau', \ \mathrm{sh}(\alpha \ominus (\beta \oplus \tau)) = \lambda, \ \mathrm{sh}(\alpha \ominus (\beta \oplus \tau')) = \mu.$$

Now, within τ, using further $\oplus$ and $\ominus$ decompositions, i.e. choosing arbitrary γ and δ of the indicated shapes and θ as shown in the remainder of Figure 1, reduces the problem to that solved in the corollary above. □

The combination of Theorem 7 and Corollary 2 provides the desired characterisation of Young classes.

Theorem 8. *A permutation class $\mathcal{C}$ is a Young class if and only if there is some downwards closed set D in the double-domination order of partitions and $\mathcal{C} = \{\pi : \mathrm{sh}(\pi) \in D\}$.*

Proof. By definition, $\mathcal{C}$ must be a union of shape equivalence classes, over a set D of shapes. To see that it is necessary that D be a down set in the double domination order suppose that $\mu \in D$. By Theorem 7, if $\lambda \preceq \mu$ has $|\lambda| + 1 = |\mu|$, there are two permutations σ and π with $\mathrm{sh}(\sigma) = \lambda$, $\mathrm{sh}(\pi) = \mu$ and $\sigma \preceq \pi$. As $\pi \in \mathcal{C}$ we must have $\sigma \in \mathcal{C}$ and hence $\lambda \in D$. However, by Proposition 3, the full double domination relation is the transitive closure of these single element covers. So, D must be down closed.

On the other hand, Corollary 2 says precisely that if D is down closed, then so is $\mathcal{C}$. □

Note that for any partition λ there exists a hook, i.e. a partition of the form $(n, 1^k)$, such that $\lambda \preceq (n, 1^k)$ (essentially we can replace each element not in the first row or column by a pair of elements, one in the first row and one in the first column.) It follows that in any proper Young class there are bounds, say a and d such that every permutation in the class either has no increasing subsequence of length a or no decreasing subsequence of length d. So every proper Young class is contained in the union of two of the prototypical Young classes already described by Schensted's theorem.

5. A special case

In general the arguments of the preceding section do not permit us to find the basis of a Young class. We can recognise the minimal shapes in its complement, but the union of the permutations of such shapes might well not be an antichain. More seriously, we might well need to include some elements of non-minimal shape as the example of the permutation class consisting of all permutations whose shapes do not doubly dominate $(2, 2, 2)$ shows (see the discussion following Corollary 2).

However, for a particular collection of classes, closely related to the prototypical Young classes we can both establish that they are Young classes, and determine a basis.

Let δ_k be the decreasing permutation of length k. For any permutation class $\mathcal{C}$ and $j \geq 0$ the permutation class $\mathcal{C}^{+j}$ consists of the set of all permutations that contain a subsequence omitting at most j elements which belongs to $\mathcal{C}$.

Theorem 9. *For any non-negative integers k and j, the class $\mathrm{Av}(\delta_k)^{+j}$ is a Young class whose basis is given by:*

$$B(\mathrm{Av}(\delta_k)^{+j}) = \left\{ \pi \,:\, \begin{array}{l} \mathrm{sh}(\pi) \text{ has exactly } j+1 \text{ cells in rows } k \text{ and} \\ \text{higher, and the first } k \text{ rows of } \mathrm{sh}(\pi) \text{ all have} \\ \text{the same length.} \end{array} \right\}$$

Proof. The characterization of the class is straight forward. If $\pi \in \mathrm{Av}(\delta_k)^{+j}$ has length n, then π contains a sub permutation of length $n-j$ in $\mathrm{Av}(\delta_k)$. In particular, the longest subsequence of π that can be written as a union of $k-1$ increasing subsequences has length at least $n-j$. Therefore, the tableau of π has at least $n-j$ cells in its first $k-1$ rows. The converse is equally immediate – if π has such a tableau then by Greene's theorem is has a subpermutation of length at least $n-j$ which can be written as a union of $k-1$ increasing permutations, and hence it belongs to $\mathrm{Av}(\delta_k)^{+j}$.

Now consider an arbitrary permutation $\theta \in B(\mathrm{Av}(\delta_k)^{+j})$. Suppose that its tableau has shape λ. Then λ must contain more than j cells in rows k and higher. Let c be the minimum number of columns of λ which contain more than j cells in rows k and higher. Note that row k of λ must contain at least c cells. Take a subpermutation θ' of θ which is the union of c descending sequences and is as long as possible. Consider now the tableau of θ'. This cannot have more than c cells in any row and so must contain more than j cells in rows k and higher. In particular, $\theta' \notin \mathrm{Av}(\delta_k)^{+j}$. But, as θ was a basis element, this implies $\theta = \theta'$. So, the first k rows of θ form a rectangle. Furthermore, θ cannot have more than $j+1$ elements in rows k or higher. So the tableau of θ must be of the specified form.

Let θ be a permutation of the form specified. Then clearly $\theta \notin \mathrm{Av}(\delta_k)^{+j}$. However, if we delete any single element of θ we can change the length of the maximal subsequence of θ that can be written as a union of k increasing sequences by at most 1. In particular, we cannot decrease the length of any of the first $k-1$ rows. For, to do so, would move at least two cells into earlier columns, leaving more cells in those columns than were present in θ, thereby creating a contradiction in that we would obtain a subpermutation of θ having a longer subsequence that is the union of some fixed number of decreasing sequences than θ had. So, any deletion from θ leaves a permutation belonging to $\mathrm{Av}(\delta_k)^{+j}$ and thus θ is a basis element. □

6. Conclusions and Open Problems

We have considered classes of permutations obtained by taking either all or none of the permutations of each shape. The general relationship between the involvement relationship for permutations and their shapes seems to be somewhat obscure. However, by restricting attention to these Young classes we were able to demonstrate a close connection with the double domination order for partitions.

More detailed connections (e.g. to describe the bases of these classes) seem to be difficult to achieve except in cases closely related to the prototypical Young classes provided by Schensted's theorem.

The most obvious remaining open question is:

> Give necessary and sufficient conditions on partitions $\lambda \preceq \mu$ which guarantee the existence of permutations σ and π with $\mathrm{sh}(\sigma) = \lambda$, $\mathrm{sh}(\pi) = \mu$ and $\sigma \preceq \pi$.

The example following Corollary 2 would seem to suggest that such conditions would involve a strengthening or refinement of Greene's theorem, but the details remain elusive.

7. Acknowledgments

The author would like to thank the University of Otago Theory Group (Mike Atkinson, Robert Aldred, Dennis McCaughan) who inspired the preliminary stages of this work in early 2009 and the organisers of the Permutation Patterns 2010 conference.

References

[1] Ron M. Adin and Yuval Roichman, *Shape avoiding permutations*, J. Combin. Theory Ser. A **97** (2002), no. 1, 162–176, DOI 10.1006/jcta.2001.3202. MR **1879132 (2002j:**05001)

[2] Curtis Greene, *An extension of Schensted's theorem*, Advances in Math. **14** (1974), 254–265. MR 0354395 (50 #6874)

[3] Greta Panova, *Separable permutations, Robinson-Schensted and shortest containing supersequences*, Permutation Patterns (Dartmouth College, 2010), available at `http://www.math.dartmouth.edu/~pp2010/abstracts/Panova.pdf`.

[4] Bruce E. Sagan, *The symmetric group*, 2nd ed., Graduate Texts in Mathematics, vol. 203, Springer-Verlag, New York, 2001. Representations, combinatorial algorithms, and symmetric functions. MR **1824028 (2001m:**05261)

[5] C. Schensted, *Longest increasing and decreasing subsequences*, Canad. J. Math. **13** (1961), 179–191. MR 0121305 (22 #12047)

[6] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR **1676282 (2000k:**05026)

[7] Mark Tiefenbruck, *231-avoiding permutations and the Schensted correspondence*, Permutation Patterns (Dartmouth College, 2010), available at `http://www.math.dartmouth.edu/~pp2010/abstracts/Tiefenbruck.pdf`.

Department of Computer Science, University of Otago, Dunedin, New Zealand

E-mail address: `malbert@cs.otago.ac.nz`