Department of Computer Science,
University of Otago

Technical Report OUCS-2003-02

Proceedings, Permutation Patterns 2003

Editor: **Michael H. Albert**

Department of Computer Science

Department of Computer Science,
University of Otago, PO Box 56, Dunedin, Otago, New Zealand

http://www.cs.otago.ac.nz/trseries/

Proceedings
Permutation Patterns 2003
University of Otago
Dunedin, New Zealand

Editor: Michael H. Albert

Contents

Introduction

Welcome to PP2003, the first conference devoted exclusively to the study of permutation patterns and related results. We thank you all for coming such great distances and hope that you will enjoy your stay.

We would like to thank the New Zealand Institue of Mathematics and its Applications (NZIMA), Otago University, the department of Computer Science, and the department of Mathematics and Statistics for having provided support to the conference in various ways.

These proceedings are intended as a set of abstracts and extended abstracts. The formatting of some of the submissions has been changed slightly to provide a (somewhat) uniform style. Many of the papers (and others) will appear as full versions in the special issue of the Electronic Journal of Combinatorics on Permutation Patterns.

List of participants

Michael Albert	malbert@cs.otago.ac.nz
Robert Aldred	raldred@maths.otago.ac.nz
Mike Atkinson	mike@cs.otago.ac.nz
Miklós Bóna	bona@math.ufl.edu
Alexander Burstein	burstein@gwu.edu
Anders Claesson	claesson@math.chalmers.se
Anant Godbole	godbolea@access.etsu.edu
Chris Handley	chandley@cs.otago.ac.nz
Peter Hästö	hasto@cc.helsinki.fi
Derek Holton	dholton@maths.otago.ac.nz
Aaron Jaggard	adj@math.upenn.edu
Ivo Klemes	klemes@math.mcgill.ca
Zvi Kohavi	kphavi@cs.technion.ac.il
Xavier Munoz	xml@mat.upc.es
Maximillian Murphy	maliculinminor@hotmail.com
Yuri Rabinovich	yuri@cs.haifa.ac.il
Astrid Reifegerste	reifegerste@math.uni-hannover.de
Bruce Sagan	sagan@math.msu.edu
Einar Steingrimmson	einar@math.chalmers.se
Hans van Ditmarsch	hans@cs.otago.ac.nz
Vincent Vatter	vatter@math.rutgers.edu
Julian West	westj@mala.bc.ca
Herbert Wilf	wilf@math.upenn.edu

Simple permutations, partial well-order, and enumeration

M. H. Albert
M. D. Atkinson
Department of Computer Science, University of Otago

1 Introduction

A class X of permutations is said to be *closed* if whenever $\sigma \in X$ and τ is a subpermutation of σ (a pattern within σ) then $\tau \in X$ also. Closed classes of permutations are precisely the classes that are defined by pattern restrictions. We often want to answer the following three questions about such classes:

1. Is there an efficient recognition algorithm for deciding when a permutation belongs to the class?
2. Can the class be defined by a *finite* set of pattern restrictions?
3. How many permutations of length n does the class contain?

The first question is NP-complete for some classes [5, 6]. The second question usually has a negative answer [1]. The third question is, in general, so difficult that we are often content with partial answers (such as confirming the Wilf-Stanley conjecture). Despite this pessimistic summary many classes that arise in practice often have positive or computable answers for all three questions. In this paper we examine some conditions which ensure that a class of permutations is tractable from this point of view.

A *component* of a permutation $\sigma = s_1 \cdots s_n$ is a segment $s_i \cdots s_j$ with the property that the set $\{s_i, \ldots, s_j\}$ is a set of consecutive integers (a *range*). A component is trivial if it is of length 1 or n. If σ has no non-trivial component it is said to be *simple*. For example, 58317462 is simple but 61835247 is not simple because of its component 3524.

We shall be concerned with closed classes of permutations which contain only finitely many simple permutations. For these classes we can establish the following:

1. There is a polynomial time algorithm to determine membership.
2. The class is defined by a finite set of pattern restrictions.
3. Let s_n be the number of permutations of length n in the class; then the ordinary generating function $\sum_{n=1}^{\infty} s_n x^n$ is algebraic.

Proofs of these results are constructive and in some special cases even stronger results can be obtained. In the next two sections we sketch the proofs of the second and third of these results (the first follows from the second); and in the final section we give an asymptotic result for the number of simple permutations of length n. The results of section two have also been obtained by Murphy [4]. Because of the constructive nature of the methods it is also possible to extend some of the results to some classes which contain an infinite family of simple permutations.

2 A finite number of restrictions

We begin with a technical result about simple permutations.

Proposition 2.1. *Every simple permutation σ of length $n > 2$ has a simple subpermutation of length $n-1$ or $n-2$. If it has no simple subpermutation of length $n-1$ then $n = 2m$ is even and $\sigma = 2, 4, \ldots, 2m, 1, 3, \ldots, 2m-1$ or one of three permutations obtained from this by symmetry.*

Proof. A proof of this may be found in [4]. A shorter proof will appear in the full version of this paper. □

Now let $F = \{\phi_1, \ldots, \phi_m\}$ be a finite set of simple permutations. To avoid trivialities we assume that any simple subpermutation of a permutation in F also lies in F. Consider the largest closed class L whose set of simple permutations is F precisely; L certainly exists since the union of two closed classes is also closed.

Lemma 2.2. *L is defined by a finite set of pattern restrictions.*

Proof. It is easy to see that a sufficient set of pattern restrictions is provided by those permutations that are minimal in the pattern containment order subject to not belonging to L (in the terminology of [1] the basis of L). Let α be such a permutation. We shall argue that α is simple.

For a contradiction, assume that $\alpha = \beta\gamma\delta$ where γ is a non-trivial component of α. By choosing γ of minimal length we may take γ to be simple. It is easy to see that any simple subpermutation of α is either a subpermutation of $\beta g\delta$ where g is the contraction of γ to a single symbol, or is γ itself; so all such simple subpermutations are in F. But then α and all of its subpermutations could be adjoined to L without adding any simple permutations, contradicting the maximality of L with respect to this property.

Since α is simple it has, by Proposition 2.1, a simple subpermutation of length $|\alpha|-1$ or $|\alpha|-2$. By the minimality of α this subpermutation is one of the permutations of F and so $|\alpha|$ is bounded. □

The proof also shows that the minimal set of restrictions that define L is easily computable. Of course L is only one (the largest) of the closed classes whose set of simple permutations is precisely F. To get a similar result for the other classes we need a partial well-order result.

Proposition 2.3. *The class L is partially well-ordered by the pattern containment order.*

Proof. We shall give L the structure of an algebra with an operator for each $\phi \in F$. Let $\phi = f_1 f_2 \cdots f_t \in F$ and let $\xi_1, \xi_2, \ldots, \xi_t$ be any t permutations of L. Then we define

$$\phi(\xi_1, \xi_2, \ldots, \xi_t) = \xi_1' \xi_2' \cdots \xi_t'$$

where each ξ_i' is a component, is order isomorphic to ξ_i, and the relative order of the components is given by the permutation ϕ. The crucial point is that the right hand side also belongs to L for it is easy to see that this permutation can have no simple subpermutations other than those in F (and so, if it was not in L, we could adjoin it and all its subpermutations to L and contradict the maximal property of L).

It follows also that L is generated, as an algebra, by the single permutation of length 1. It is easy to check that the conditions necessary to apply Higman's theorem [2] are satisfied and therefore L is partially well-ordered. $\square$

Now we have, by Proposition 1.1 of [1],

Corollary 2.4. *Every closed class of permutations having only a finite number of simple permutations is determined by a finite set of restrictions.*

3 Algebraic generating functions

We shall outline a calculus capable of enumerating closed classes that have only a finite number of simple permutations. This calculus exploits the following way of decomposing a permutation π. Consider the maximal proper components of π. If two such components overlap then clearly their union is also a component, hence the whole of π; in this case it is easy to see that π decomposes as $\pi = \alpha\beta$ where either $\alpha < \beta$ or $\alpha > \beta$. In the more interesting case that π has no such decomposition, the maximal proper components form a partition of π and the pattern determined by the blocks of this partition define a simple permutation of length more than 2 called the *skeleton* of π.

Suppose that π is a simple permutation of length $n > 2$, and $A_1, A_2, \ldots, A_n$ are sets of permutations. Then

$$\pi(A_1, A_2, \ldots, A_n)$$

denotes the set of all permutations θ that have a wreath decomposition

$$\theta = \theta_1\, \theta_2\, \cdots\, \theta_n$$

whose pattern is π and for which $\theta \in A_i$. Notice that every permutation of this set has skeleton π. From the uniqueness of the skeleton we have

Lemma 3.1. *If π is simple of length $n > 2$ then*

$$\pi(A_1, A_2, \ldots, A_n) \cap \pi(B_1, B_2, \ldots, B_n) = \pi(A_1 \cap B_1, A_2 \cap B_2, \ldots, A_n \cap B_n)$$

To see how these ideas apply to enumerating classes with only a finite number of simple permutations we first consider again the largest class L whose simple permutations are $\{12, 21, \phi_3, \ldots, \phi_m\}$. Permutations of L either have skeleton one of $\phi_3, \ldots, \phi_m$ or the form $\alpha\beta$ with $\alpha < \beta$ or $\alpha > \beta$. The first types are permutations within $\phi_i(L, L, \ldots, L)$ and all these sets are disjoint. This gives rise to an equation for the generating function of the form

$$f(x) = \sum_{i=3}^{m} f(x)^{|\phi_i|} + p(x) + q(x)$$

Each summand on the right hand side arises from one of the forms above. We omit here the special argument that is used to handle the terms $p(x)$ and $q(x)$.

To obtain information about the generating functions of subclasses of L we shall use the main result of the previous section: these subclasses arise from L by imposing a finite number of further pattern restrictions. When we start imposing such restrictions we find that sets of the form $\pi(A_1, A_2, \ldots, A_n)$ as defined above play a crucial part. In order to get an inductive argument going, we have to analyse the effect of restricting such a set by a single permutation τ.

In general, the notation $A[\tau_1, \tau_2, \ldots, \tau_k]$ denotes the subclass of A whose permutations avoid all of $\tau_1, \tau_2, \ldots, \tau_k$. To analyse the class

$$\pi(A_1, A_2, \ldots, A_n)[\tau]$$

we define an *embedding by blocks* of τ in π to consist of a decomposition $\tau = \tau_1 \cdots \tau_l$ whose pattern σ is a subpermutation of π together with a map

$$s : \{1, 2, \ldots, l\} \longrightarrow \{1, 2, \ldots, n\}$$

expressing the subpermutation embedding. Let E be the set of all such embeddings by blocks.

If τ is a subpermutation of some element $\alpha = \alpha_1 \cdots \alpha_n$ of $\pi(A_1, A_2, \ldots, A_n)$ then there must be some embedding by blocks of τ in π such that each of the parts τ_i of the decomposition is a subpermutation of $\alpha_{s(i)}$. So the elements of $\pi(A_1, A_2, \ldots, A_n)[\tau]$ are those for which no $s \in E$ is such an embedding; hence for every $s \in E$ there is some part τ_i that is not a subpermutation of $\alpha_{s(i)}$. Therefore

$$\pi(A_1, A_2, \ldots, A_n)[\tau] = \bigcap_E \bigcup_{i=1}^{l} \pi(A_1, \ldots, A_{s(i)}[\tau_i], \ldots \ldots, A_n)$$

We may write the right hand side in disjunctive normal form (as a union of terms, each of which is an intersection of terms like $\pi(A_1, \ldots, A_{s(i)}[\tau_i], \ldots \ldots, A_n)$). These intersections, by Lemma 3.1, have the form $\pi(B_1, \ldots, \ldots, B_n)$ where each B_i is either A_i or A_i restricted by finitely many permutations.

The size of the left hand side can now be expressed, using inclusion exclusion, as a sum and difference of sizes of these $\pi(B_1, \ldots, \ldots, B_n)$ and their intersections taken several at a time. Again, by Lemma 3.1, such intersections have the same form.

In this way we can express the enumeration of $\pi(A_1, A_2, \ldots, A_n)[\tau]$ in terms of enumerations of simpler sets of the same type and this leads to the theorem

Theorem 3.2. *The generating function of every class with just a finite number of simple permutations is algebraic.*

4 Asymptotics of simple permutations

The number s_n of simple permutations of small degree can be computed by means of a straighforward recurrence, and the first few values are given by

n	2	3	4	5	6	7	8	9	10	11	12
s_n	2	0	2	6	46	338	2926	28146	298526	3454434	43286526

We also have an asymptotic result.

Theorem 4.1.

$$\lim_{n\to\infty} \frac{s_n}{n!} = \frac{1}{e^2}$$

Proof. Let S_n denote the numbers of simple permutations of length n and T_n the number of permutations with no simple component of size 2. Obviously $S_n \subseteq T_n$. It is proved in [3] that

$$\lim_{n\to\infty} \frac{|T_n|}{n!} = \frac{1}{e^2}$$

Since $T_n \setminus S_n$ consists of permutations which have a simple component of size k, with $4 \leq k < n$, it is enough to show that the number u_n permutations of this latter type is asymptotically negligible.

We can choose the position of a simple component of length k in $n-k+1$ ways, the set of consecutive values that comprise it in $n-k+1$ ways, their arrangement in s_k ways, and the arrangement of the remaining items in $(n-k)!$ ways so we have

$$\frac{u_n}{n!} \leq \sum_{k=4}^{n-1} \frac{s_k(n-k+1)^2(n-k)!}{n!}$$

For $4 \leq k \leq n-4$ we bound the kth term by

$$\begin{aligned} \frac{s_k(n-k+1)^2(n-k)!}{n!} &\leq \frac{k!\, n^2(n-k)!}{n!} \\ &\leq n^2 \binom{n}{k}^{-1} \leq n^2 \binom{n}{4}^{-1} \\ &= O(1/n^2) \end{aligned}$$

For $n-4 < k \leq n-1$ we can bound the kth term using $s_k \leq (n-1)!$ and $n-k+1 \leq 4$ so that

$$\frac{s_k(n-k+1)^2(n-k)!}{n!} = O(1/n)$$

Thus $u_n/n! = O(1/n)$ and the proof is complete. $\square$

References

[1] M. D. Atkinson, M. M. Murphy, N. Ruškuc, Partially well-ordered closed sets of permutations, Journal of Order 19 (2002), 101–113.

[2] G. Higman, Ordering by divisibility in abstract algebras, J. London Math. Soc. 2 (1952), 326–336.

[3] I. Kaplansky, The asymptotic distribution of runs of consecutive elements, Ann. Math. Statistics, 16 (1945), 200–203.

[4] M. M. Murphy, Restricted permutations, antichains, atomic classes and stack sorting, PhD thesis, University of St Andrews, 2002.

[5] W. Unger, The complexity of colouring circle graphs, Proceedings 9th Annual Symposium on Theoretical Aspects of Computer Science, 1992, Springer Lecture Notes in Computer Science 577, 389–400.

[6] W. Unger, On the k-colouring of circle graphs, Proceedings 5th Annual Symposium on Theoretical Aspects of Computer Science, 1988, Springer Lecture Notes in Computer Science 294, 61–72.

Sorting with a forklift.

M. H. Albert
M. D. Atkinson
Department of Computer Science, University of Otago

A standard analogy for explaining the operation of a stack is to speak about stacks of plates, allowing one plate to be added to, or removed from, the top of the stack. In the context of this analogy it is natural to wonder why multiple plates cannot be simultaneously removed from or added to the stack. In this talk we explore the ramifications of allowing such operations.

In particular we will consider the problem of a dishwasher and his helper. The dishwasher receives dirty plates, washes them, and adds them one at a time to a stack to be put away. The helper can remove plates from the stack, but she can move more than one plate at a time. It so happens, that all the plates are of slightly differing sizes, and her objective is to make sure that when they are placed in the cupboard, they range in order from biggest at the bottom, to smallest at the top. It is easy to see that if the dishwasher processes three dishes in order: middle, smallest, largest then his helper can easily succeed in the model given (simply waiting for the largest plate, and then moving the two smaller ones on top as a pair). However, if she is replaced by a small child who can only move a single plate at a time, then the desired order cannot be achieved.

An alternative, slightly more general, analogy provides the source of our title. We begin with a stack of boxes, called the input, labelled 1 through n in some order. We have at our disposal a powerful forklift which can remove any segment of boxes from the top of the stack, and move it to the top of another stack, the working stack. From there another forklift can move the boxes to a final, output, stack. Physical limitations, or union rules, prevent boxes being moved from the working stack to the input, or from the output to the working stack. The desired outcome is that the output should be ordered with box number 1 on top, then 2, then 3, ..., with box n at the bottom.

The problems we wish to consider in this context are:

- How should such permutations be sorted?
- Which permutations can be successfully sorted?
- How many such permutations are there?

We will also consider these questions in the restricted context where one or both of the moves allowed are of limited power – for example, say, at most three boxes at a time can be moved from the input stack to the working stack, and at most six from the working stack to the output stack.

Our main results are the following:

- In either the full or restricted contexts the class of permutations which are sortable is a class defined by a finite number of pattern restrictions.
- There is a simple algorithm for recognizing the sortable permutations (i.e. for operating the forklifts when sorting is possible).

- In any restricted context, the generating function for the class of sortable permutations is algebraic and, in principle, can be computed.

The results in the case of the dishwasher problem are particularly interesting. In this case, with the most limited possible helper, the class is of course one of the Catalan classes, with exponential constant 4 in its growth rate. With a helper of unlimited power the enumeration sequence is:

$$1, 1, 2, 6, 21, 79, 311, 1265, 5275, 22431, 96900, 424068, \ldots$$

the binomial transform of Fine's sequence, and has exponential constant 5. Both these generating functions satisfy quadratic equations. However, for helpers who can lift k dishes where $2 \leq k < \infty$, the degree of the polynomial satisfied by the corresponding generating function is $k+1$, and the difference between 5 and its radius of convergence is geometrically decreasing (asymptotically, by a factor of 3).

A survey of stack-sortable permutations

Miklós Bóna
Department of Mathematics
University of Florida
Gainesville FL 32611

Abstract

We discuss various recent developments concerning stack-sortable permutations. We mention results connected to unimodality, symmetry, and simplicial complexes. Conjectures on log-concavity and enumeration are included

1 Introduction

In what follows, permutations of length n will be called n-permutations. The stack-sorting operation s can be defined on the set of all n-permutations as follows. Let $p = LnR$ be an n-permutation, with L and R respectively denoting its subword before and after the maximal entry. Let $s(p) = s(L)s(R)n$, where L and R are defined recursively by this same rule. For a nonrecursive, algorithmic definition, or the origin of the notion see [8], [10].

A permutation p is called t-stack sortable if $s^t(p)$ is the identity permutation. The stack-sorting operation, and 2-stack sortable permutations, were the subject of numerous research efforts during the past decade, and connections between this field and labeled trees [9], Young tableaux [7], and planar maps have been found.

The set of 1-stack sortable n-permutations is easy to characterize by the following notion of pattern avoidance. Let $q = (q_1, q_2, \ldots, q_k)$ be a k-permutation and let $p = (p_1, p_2, \ldots, p_n)$ be an n-permutation. We say that p contains a q-subsequence if there exists $1 \leq i_{q_1} < i_{q_2} < \ldots < i_{q_k} \leq n$ such that $p_{i_1} < p_{i_2} < \ldots < p_{i_k}$. We say that p avoids q if p contains no q-subsequence. For example, p avoids 231 if it cannot be written as $\cdots, a, \cdots, b, \cdots c, \cdots$ so that $c < a < b$. It is easy to show [8] [10] that a permutation is 1-stack sortable if and only if it avoids the pattern 231. In particular, the number of 1-stack sortable permutations is therefore C_n, the nth Catalan number.

The set of 2-stack sortable permutations is much more complex. For example, it is not true that a subword of a 2-stack sortable permutation is always 2-stack sortable; for 35241 is 2-stack sortable, while its substring 3241 is not. So in general, t-stack sortability cannot be described by regular pattern avoidance. Similarly, it is far more difficult to find a formula for the number W_n of these permutations [9], [5], [11] than for that of 1-stack sortable ones.

2 Symmetry and Unimodality

The exact numbers $W_t(n,k)$ of t-stack sortable n-permutations with k descents are known if $t = 1$ or $t = 2$, and show several interesting properties.

As 1-stack sortable permutations are the 231-avoiding ones, it is well-known that the numbers $W_1(n,k)$ will be the famous Narayana numbers, that is,

$$W_1(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

In particular, for any fixed n, the sequence $\{W_1(n,k)\}$, $0 \leq k \leq n-1$ is symmetric. This symmetry is further explained by present author and R. Simion, who proved (up to a trivial symmetry) that there are as many 1-stack sortable n-permutations with descent set $S \subseteq \{1,2,\cdots n-1\}$ as there are with descent set $S*$, where $S*$ is the *reverse complement* of S, that is, $i \in S*$ if and only if $n - i \notin S$.

If $t = 2$, then determining the numbers $W_2(n,k)$ is much more difficult. Constructing a bijection with nonseparable planar maps, it can be shown [9] that

$$W_2(n,k) = \frac{(n+k)!(2n-k-1)!}{(k+1)!(n-k)!(2k+1)!(2n-2k-1)!}.$$

In particular, we get again that for any fixed n, the sequence $\{W_2(n,k)\}$, $0 \leq k \leq n-1$ is symmetric. This raises the question whether the sequence $W_t(n,k)$, $0 \leq k \leq n-1$ is symmetric for *any* fixed t and any fixed n.

We will show that this sequence is in fact symmetric, and even unimodal for all fixed t and n. We conjecture that the sequence is also log-concave, and that its generating function has real zeros only.

3 A Simplicial Complex of 2-stack sortable permutations

The polynomials $W_{n,t}(x)$ are generalizations of the Eulerian polynomials. Let us find out which properties of the Eulerian polynomials are preserved by this generalization.

Recently, V. Gasharov constructively proved the following interesting result [6] that has been proved before by F. Brenti [3] by other means, and discussed as a special case of a more general setup (Coxeter groups, instead of just the symmetric group).

Denote by $[n]$ the set $\{1, 2, ..., n\}$.

Theorem 3.1. *There exists a simplicial complex whose $(k-1)$-dimensional faces correspond to permutations of $[n]$ with k descents.*

Gasharov's constructive proof raises the following question. Is it true that for any fixed n, there a simplicial complex whose $(k-1)$-dimensional faces correspond to *t-stack sortable* permutations of $[n]$ with k descents (or ascents)? In other words, are the polynomials $W_{n,t}(x)$ Hilbert series?

We answer this question for each n in the affirmative for the easy case of $t = 1$, and for the more interesting case of $t = 2$. While these questions could be answered in other ways, our proofs will be constructive. In this case, this is an important difference. Indeed, the reason that makes a different, computational approach possible when $t = 1$ or $t = 2$ is that there are formulae [9] [1], for the numbers $W_1(n,k)$, and $W_2(n,k)$. Once all these numbers are known, there is a numerical sufficient and necessary condition [3] for them to form the f-vector of a simplicial complex. However, if these numbers are not known, as it is the case when $t > 2$, that approach will not work. Present author conjectures that the answer to this question is in the affirmative for all t.

4 The case of $t = 1$

Theorem 4.1. *There exists a simplicial complex S whose $k-1$-dimensional faces correspond to stack sortable permutations with k ascents.*

Proof. As we mentioned, a permutation is stack sortable if and only if it avoids the pattern 231. On the other hand, 231-avoiding n-permutations are in bijection with northeastern lattice paths from $(0,0)$ to (n,n) that never go above the main diagonal. The following refinement of this fact is well-known, but we include it for self-containment.

Lemma 4.2. *There is a bijection r from the set of 231-avoiding n-permutations with k ascents onto that of NE lattice paths from $(0,0)$ to (n,n) that never go above the main diagonal and have k north-to-east turns.*

Proof. Let $p = p_1p_2\cdots p_n$ be a 231-avoiding n-permutation with k ascents. If the entry n of p is not in the first position, then everything on the left of n must be smaller than everything on the right of n. That is, if $n = p_i$, then the string $p' = p_1p_2\cdots p_{i-1}$ forms a 231-avoiding permutation on $[i-1]$, while the string $p'' = p_ip_{i+1}\cdots p_n$ forms a 231-avoiding permutation on $\{i, i+1, \cdots, n\}$. Thus we can define $r(p) = L$ as the lattice path that is a concatenation of L_1 and L_2, where $L_1 = r(p')$ and $L_2 = r(p'')$ as defined recursively by this same algorithm. (The path L_1 goes from $(0,0)$ to $(i-1,i-1)$, while L_2 goes from $(i-1,i-1)$ to (n,n)).

If $n = p_1$, then the first step of $r(p) = L$ is to the east, and the last step of L is to the north. We still have to define the part of L that goes from $(1,0)$ to $(n,n-1)$. That part is set to be $r(p_2p_3\cdots p_n)$, defined again recursively by this same algorithm.

It is straightforward to check (by induction, or otherwise), that the map r we constructed is a bijection with the desired property. $\diamond$

Now the proof of Theorem 4.1 is straightforward. Take any northeastern lattice path L that has k north-to-east turns. Denote $t_1, \cdots, t_k$ the positions of these turns. Here the t_i are points in $\mathbf{N^2}$, and $t_1 < \cdots < t_k$ in the usual (coordinate-wise) ordering of $\mathbf{N^2}$.

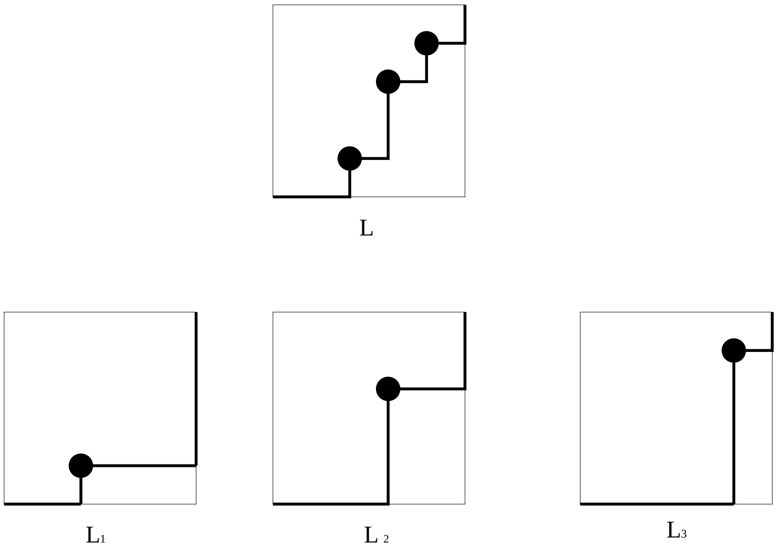


Figure 1: Decomposing a lattice path.

It is clear that the set of its north-to-east turns completely determines L, so f is an injection. On the other hand, f is not a surjection onto the set of all ordered k-tuples of subdiagonal NE paths from $(0,0)$ to (n,n). Indeed, for $(L_1, L_2, \cdots L_k)$ to have a preimage, all the k north-to-east turns contained in the L_i must be in different rows, and in different columns. Moreover, the subposet of $\mathbf{N^2}$ that is induced by the elements $t_1, \cdots, t_k$ has to be a chain. In other words, if $i < j$, and $t_i = (a, b)$, and $t_j = (c, d)$, then $a < c$ and $b < d$ have to hold.

We define $\mathcal{L}$ to be the set of all subdiagonal NE paths from $(0,0)$ to (n,n) having one NE turn. Let Δ_1 be the simplicial complex of all ordered subsets $(L_1, L_2, \cdots L_m)$ of $\mathcal{L}$ that have a preimage by f. Then the above discussion shows that the $k-1$ dimensional faces of this complex are precisely the subdiagonal NE lattice paths $(0,0)$ to (n,n) having k NE turns. $\diamond$

We point out that the definition of simplicial complexes uses *sets* of nodes, not k-tuples, or in other words, *ordered sets*. This did not cause a problem, however, for each set of k NE turns could have at most one ordering that belonged to our simplicial complex. This phenomenon will occur in the next section, too.

5 The case of $t = 2$

In this section, we prove the following analogue of Theorem 3.1 for 2-stack sortable permutations.

Theorem 5.1. *There exists a simplicial complex S whose $k-1$-dimensional faces correspond to stack 2-sortable permutations with k ascents.*

In general, 2-stack sortable permutations turn out to be much more difficult to handle than stack sortable permutations, and this particular problem is no exception. The extra layer of difficulty lies in the representation of these permutations by other objects. The representation we will use needs the following definition.

Definition 5.2. *[4] [9] A rooted plane tree with positive integer labels $l(v)$ on each of its nodes v is called a $\beta(1,0)$-tree if it satisfies the following conditions:*

- *if v is a leaf, then $l(v) = 1$,*
- *if v is the root and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) = \sum_{i=1}^{k} l(v_k)$,*
- *if v is an internal node (that is, not the root, and not a leaf), and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) \leq \sum_{i=1}^{k} l(v_k)$.*

Note that this implies that no node can have a larger label than the total number of its descendents. See Figure 2 for an example.

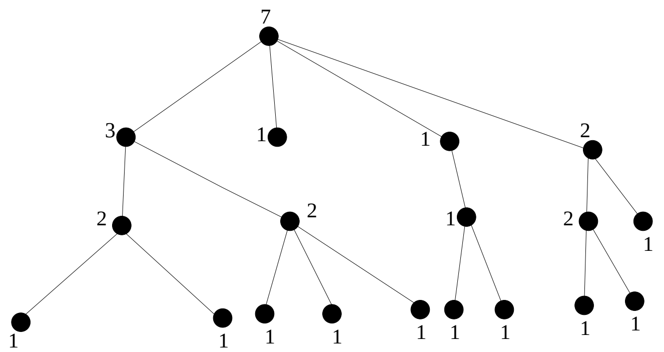


Figure 2: A $\beta(1,0)$-tree .

The relevance of $\beta(1,0)$-trees to our problem is revealed by the following Theorem, which is quite difficult to prove.

Theorem 5.3. *[4] [9] There exists a bijection b from the set of all $\beta(1,0)$-trees on $n+1$ nodes onto that of all 2-stack sortable n-permutations so that if a $\beta(1,0)$-tree T has k internal nodes, then $b(k)$ has k ascents.*

Denote $D_{n+1,k}^{\beta(1,0)}$ the set of all $\beta(1,0)$-trees on $n+1$ nodes having k internal nodes. Our plan is as follows. To each $\beta(1,0)$-tree $T \in D_{n+1,k}^{\beta(1,0)}$ we will associate a k-tuple $(T_1, T_2, \cdots, T_k) \in [D_{n+1,1}^{\beta(1,0)}]^k$ of $\beta(1,0)$-trees . Again, we will do it in an injective way.

We have to specify the order in which we will treat the k internal nodes of T. We define an order call *postorder* on the set of all nodes of T; the restriction of that order to the set of internal nodes will then tell us in which order to treat the internal nodes.

Postorder is defined as follows. For every node V, first we read the subtrees of the children of V from left to right, then V itself. The subtrees of the children of V are read recursively, according to the same rule. This rule linearly orders all nodes of

T, and in particular, turns our set $\{V_1, \cdots, V_k\}$ of internal nodes into the k-tuple $(V_1, \cdots, V_k)$ of internal nodes.

Let $i \in [k]$ and let V_i be the ith internal node of our $\beta(1,0)$-tree T. Let V_i have d_i descendents, excluding itself. Moreover, denote l_i the number of nodes of T that precede V_i in the postorder reading of T. Similarly, denote by r_i the nodes of T follow V_i in the postorder reading of T.

Then we define T_i as the unique $\beta(1,0)$-tree with one internal node Z_i so that Z_i has d_i descendants, and the root of T_i has l_i leaf-children on the left of Z_i and r_i leaf-children on the right of Z_i. The only node whose label has to be defined is the only internal node Z_i, and we set $label_{T_i}(Z_i) = label_T(V_i)$.

We show that we can indeed always set $label_{T_i}(Z_i) = label_T(V_i)$, that is, $label_T(V_i)$ is never too big for the label of Z_i. Indeed, Z_i has d_i children, all leafs, so any positive integer at most as large as d_i is a valid choice for the label of Z_i. On the other hand, V_i has d_i descendents in T, so $label_T(V_i) \leq d_i$, and therefore $label_T(V_i)$ is indeed a valid choice for $label_{T_i}(Z_i)$.

Now we define our decomposition map, $h(T) = (T_1, T_2, \cdots, T_k)$. See Figure 3 for an example of this map.

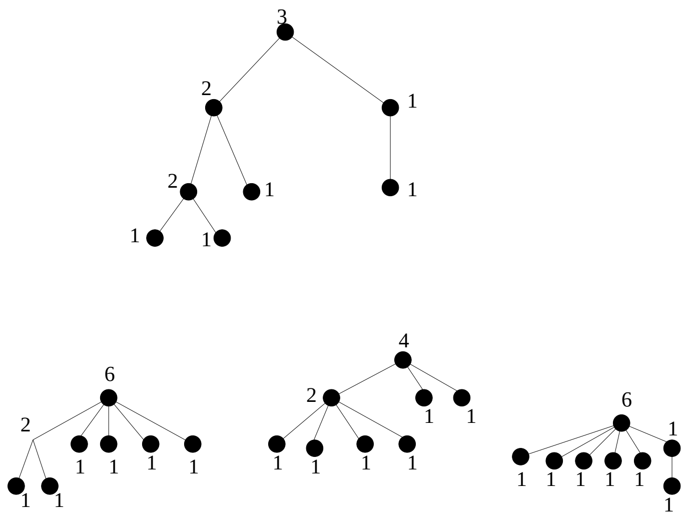


Figure 3: Decomposing a $\beta(1,0)$-tree .

Lemma 5.4. *The map* $h : D^{\beta(1,0)}_{n+1,k} \rightarrow [D^{\beta(1,0)}_{n+1,1}]^k$ *defined by* $h(T) = (T_1, T_2, \cdots, T_k)$ *is an injection.*

References

[1] M. Bóna, Symmetry and Unimodality in Stack sortable permutations, *J. Combin. Theory Ser. A*, to appear.

[2] M. Bousquet-Melou, Multi-statistic enumeration of two-stack sortable permutations, *Electronic J. Combinatorics* **5** (1998) R21.

[3] F. Brenti, Hilbert polynomials in combinatorics. *J. Algebraic Combin.* **7** (1998), no. 2, 127–156.

[4] R. Cori, B. Jacquard, G. Schaeffer, Description trees for some families of planar maps, *Proceedings of the 9th Conference on Formal Power Series and Algebraic Combinatorics*, (Vienna, 1997) 196-208.

[5] S. Dulucq. S. Gire, J. West, Permutations with forbidden subsequences and nonseparable planar maps. Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993), *Discrete Math*, **153** (1996), 85-103.

[6] V. Gasharov, On the Neggers-Stanley conjecture and the Eulerian polynomials, *J. Combin. Theory Ser. A* **82** (1998), 134–146.

[7] O. Guibert, Stack words, standard Young tableaux, permutations with forbidden subsequences and planar maps. Formal power series and algebraic combinatorics (Minneapolis, MN, 1996). *Discrete Math.* **210** (2000), no. 1-3, 71–85.

[8] D. E. Knuth, "The art of computer programming," volume 3, Sorting and searching, Addison-Wesley, Reading, Massachusetts, 1973.

[9] B. Jacquard, G. Schaeffer, A bijective census of nonseparable planar maps. *J. Combin. Theory Ser. A* **83** (1998), no. 1, 1-20.

[10] J. West, Permutations with forbidden subsequences; and, Stack sortable permutations, PHD-thesis, Massachusetts Institute of Technology, 1990.

[11] D. Zeilberger, A proof of Julian West's conjecture that the number of two-stack-sortable permutations of length n is $2(3n)!/((n+1)!(2n+1)!)$. *Discrete Math.* **102** (1992), 85-93.

Packing patterns into words

Alexander Burstein
Department of Mathematics
Iowa State University
Ames, IA 50011-2064 USA
`burstein@math.iastate.edu`

Toufik Mansour
Department of Mathematics
Chalmers University of Technology
S-412 96 Göteborg, Sweden
`toufik@math.chalmers.se`

Abstract

We define the packing density on words and find the packing densities of several types of patterns with repeated letters allowed.

A string 213322 contains three subsequences $233, 133, 122$ each of which is *order-isomorphic* (or simply *isomorphic*) to the string 122, i.e. ordered in the same way as 122. In this situation we call the string 122 a *pattern*.

Herb Wilf first proposed the systematic study of pattern containment in his 1992 address to the SIAM meeting on Discrete Mathematics. However, several earlier results on pattern containment exist, for example, those by Knuth [11] and Tarjan [15].

Most results on pattern containment actually deal with *pattern avoidance*, in other words, enumerate or consider properties of strings over a totally ordered alphabet which avoid a given pattern or set of patterns. Knuth [11] found that, for any $\pi \in S_3$, the number of n-permutations avoiding π is C_n, the nth Catalan number. Later, Simion and Schmidt [13] determined the number the number of permutations in S_n simultaneously avoiding any given set of patterns $\Pi \subseteq S_3$. Burstein [4] extended this result to the number of strings with repeated letters avoiding any set of patterns $\Pi \subseteq S_3$. Burstein and Mansour [5] considered forbidden patterns with repeated letters.

There is considerably less research on other aspects of pattern containment, specifically, on packing patterns into strings over a totally ordered alphabet (but see [1, 3, 12, 14]). In fact, all pattern packing except the one in [14] (later generalized in [1]) dealt with packing permutation patterns into permutations (i.e. strings without repeated letters). In this paper, we generalize the packing statistics and results to patterns over strings with repeated letters and relate them to the corresponding results on permutations.

1 Preliminaries

Let $[k] = \{1, 2, \ldots, k\}$ be our canonical totally ordered alphabet on k letters, and consider the set $[k]^n$ of n-letter words over $[k]$. We say that a pattern $\pi \in [l]^m$

occurs in $\sigma \in [k]^n$, or that σ *contains* the pattern π, if there is a subsequence of σ order-isomorphic to π.

Given a word $\sigma \in [k]^n$ and a set of patterns $\Pi \subseteq [l]^m$, let $\nu(\Pi, \sigma)$ be the total number of occurrences of patterns in Π (Π-patterns, for short) in σ. Obviously, the largest possible number of Π-occurrences in σ is $\binom{n}{m}$, when each subsequence of length m of σ is an occurrence of a Π-pattern. Define

$$\begin{aligned}
\mu(\Pi, k, n) &= \max\{\, \nu(\Pi, \sigma) \mid \sigma \in [k]^n \},\\
d(\Pi, \sigma) &= \frac{\nu(\Pi, \sigma)}{\binom{n}{m}},\\
\delta(\Pi, k, n) &= \frac{\mu(\Pi, k, n)}{\binom{n}{m}} = \max\{\, d(\Pi, \sigma) \mid \sigma \in [k]^n \},
\end{aligned}$$

respectively, the maximum number of Π-patterns in a word in $[k]^n$, the probability that a subsequence of σ of length m is an occurrence of a Π-pattern, and the maximum such probability over words in $[k]^n$. We want to consider the asymptotic behavior of $\delta(\Pi, k, n)$ as $n \to \infty$ and $k \to \infty$.

Proposition 1.1. *If $n > m$, then $\delta(\Pi, k, n) \leq \delta(\Pi, k, n-1)$ and $\delta(\Pi, k, n) \geq \delta(\Pi, k-1, n)$.*

Proof. The proof of Proposition 1.1 in [1] also applies to the first inequality in our proposition as well, since possible repetition of letters is irrelevant here. To see that the second inequality is true, note that increasing k, i.e. allowing more letters in our alphabet, can only increase $\mu(\Pi, k, n)$, and hence, $\delta(\Pi, k, n)$. □

The greatest possible number of distinct letters in a word σ of length n is n, which implies that $\mu(\Pi, k, n) = \mu(\Pi, n, n)$ for $k \geq n$, and hence, $\delta(\Pi, k, n) = \delta(\Pi, n, n)$ for $k \geq n$. Therefore,

$$\delta(\Pi, n, n) = \lim_{k\to\infty} \delta(\Pi, k, n).$$

We also have $\delta(\Pi, n, n) = \delta(\Pi, n+1, n) \geq \delta(\Pi, n+1, n+1)$, so $\delta(\Pi, n, n)$ is nonincreasing and nonnegative, so there exists

$$\delta(\Pi) = \lim_{n\to\infty} \delta(\Pi, n, n) = \lim_{n\to\infty} \lim_{k\to\infty} \delta(\Pi, k, n).$$

We call $\delta(\Pi)$ the *packing density* of Π.

Obviously, there are two double limits. Since $0 \leq \delta(\Pi, k, n) \leq 1$, it immediately follows that there exists

$$\delta(\Pi, k) = \lim_{n\to\infty} \delta(\Pi, k, n) \in [0, 1]$$

and that $\{\delta(\Pi, k) \mid k \in \mathbb{N}\}$ is nondecreasing as $k \to \infty$. Hence, there exists

$$\delta'(\Pi) = \lim_{k\to\infty} \delta(\Pi, k) = \lim_{k\to\infty} \lim_{n\to\infty} \delta(\Pi, k, n).$$

It is easy to see that $\delta'(\Pi) \leq \delta(\Pi)$. Naturally, one wishes to determine when $\delta'(\Pi) = \delta(\Pi)$. In this paper, we will provide a sufficient condition for this equality.

The set $[k]^n$ is finite, so for each k and n, there is a string $\sigma(\Pi, k, n) \in [k]^n$ such that $d(\Pi, \sigma(\Pi, k, n)) = \delta(\Pi, k, n)$. To find $\delta(\Pi)$, we will need to find $\delta(\Pi, k, n)$, hence maximal Π-containing permutations $\sigma(\Pi, k, n)$ are of interest to us, especially, their asymptotic shape as $n \to \infty$ and $k \to \infty$.

Example 1.2. *Let $\Pi = \{c_m\}$, where c_m is a constant string of m 1's. Then, clearly, $\sigma(\Pi, k, n) = c_n$ and $d(c_m, c_n) = 1$ for $n \geq m$, so $\delta(c_m, k, n) = 1$ for $n \geq m$, and hence $\delta'(c_m) = \delta(c_m) = 1$ for any $m \geq 1$.*

Example 1.3. *Let $\Pi = \{id_m\}$, where id_m is the identity permutation of S_m. Then $\sigma(id_m, n, n) = id_n$, so $d(id_m, id_n) = 1$, $\delta(id_m, n, n) = 1$ and $\delta(id_m) = 1$.*

Determining $\delta'(id_m)$ is a bit harder. Consider a permutation $\tau \in [k]^n$. Deleting the all 1's of σ and inserting them at the left end of τ can only increase the number of occurrences of id_m. Call the resulting permutation τ_1. Then $d(id_m, \tau) \leq d(id_m, \tau_1)$. Similarly, deleting all 2's of τ_1 and inserting them immediately after the (initial) block of 1's can only increase the the number of occurrences of id_m. Iterating this procedure k times, we see that $\sigma(id_m, k, n)$ must be an nondecreasing string of digits in $[k]$. Let n_i be the number of digits i in $\sigma(id_m, k, n)$, then $\mu(id_m, k, n) = \nu(id_m, \sigma(id_m, k, n)) = n_1 n_2 \ldots n_k$ and $n_1 + n_2 + \cdots + n_k = n$. To maximize the above product we need $n_1 = n_2 = \cdots = n_k = \frac{n}{k}$. (More exactly, [12] shows that we should choose for n_i's to be such integers that $|n_i - \frac{n}{k}| < 1$ and $|n_1 + \cdots + n_r - \frac{rn}{k}| < 1$ for each $r = 1, 2, \ldots, k$.) It follows that

$$\delta(id_m, k, n) \sim \frac{\binom{k}{m}\left(\frac{n}{k}\right)^m}{\binom{n}{m}}$$

(where $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$), so $\delta(id_m, k) = \frac{\binom{k}{m}}{k^m}$, and thus $\delta'(id_m) = 1$ as expected.

Packing density was initially defined for patterns in permutations. Therefore, we must show that the packing density on permutations agrees with the packing density on words.

Theorem 1.4. *Let $\Pi \subseteq S_m$ be a set of permutation patterns, then*

$$\delta(\Pi) = \lim_{n\to\infty} \frac{\max\{\, \nu(\Pi, \sigma) \mid \sigma \in S_n\}}{\binom{n}{m}},$$

i.e. the packing density of Π on words is equal to that on permutations.

Proof. It is enough to prove that

$$\mu(\Pi, n, n) = \max\{\, \nu(\Pi, \sigma) \mid \sigma \in S_n\},$$

in other words, that there is a permutation in S_n among the maximal Π-containing words in $[n]^n$. Consider any maximal Π-containing word $\sigma \in [n]^n$. Let n_i be the multiplicity of the letter i in σ. Let i_j denote the jth occurrence of the letter i, and consider the map $f : [n]^n \to S_n$ induced by the map $i_j \mapsto \sum_{r=1}^{i} n_r - j + 1$. Since all letters of each pattern in Π are distinct, Π occurs in $f(\sigma)$ at least at the same positions Π occurs in σ, so $\nu(\Pi, f(\sigma)) \geq \nu(\Pi, \sigma)$. The rest is easy. □

Apart from computing packing densities of patterns, we would also like to determine which patterns have equal packing densities, which ones are asymptotically more packable than others, etc. For example, it is easy to see that the packing density is invariant under the usual symmetry operations on $[l]^m$: *reversal* $r : \tau(i) \to \tau(m - i + 1)$ and *complement* $c : \tau(i) \to l - \tau(i) + 1$, (packing density is also invariant under inverse $i : \tau \to \tau^{-1}$ when packing permutations into permutations). The operations r and c generate D_2, while r, c, i generate D_4. Patterns which can be obtained from each other by a sequence of symmetry operations are said to belong to the same *symmetry class*.

Example 1.5. *The symmetry class representatives of patterns in* $[3]^3$ *are*

$$123, 132, 112, 121, 111.$$

We know that $\delta(111) = 1 = \delta(123)$. *Galvin, Kleitmann and Stromquist (independently, unpublished, see chronology in [12]) showed that* $\delta(132) = 2\sqrt{3} - 3 \approx 0.4641$. *Thus, we only need to determine the packing densities of 112 and 121 to completely classify patterns of length 3.*

Price [12] extended Stromquist's results [14] to packing a single pattern $\pi = 1m(m-1)\dots 2$ and handled other single patterns such as 2143. Since we will also be concerned mostly with singleton sets of patterns $\Pi = \{\pi\}$, we will write $\delta(\pi)$ for $\delta(\{\pi\})$, etc.

Price's results deal with patterns of specific type, the so-called *layered* patterns.

Definition 1.6. *A* layered *pattern is a strictly increasing sequence of strictly decreasing substrings. These substrings are called the* layers *of* σ.

For example, $\widehat{1}\widehat{2}\widehat{3}, \widehat{1}\widehat{32}, \widehat{21}\widehat{3}, \widehat{321}$ are layered, with layers denoted by hats, while $312, 231$ are non-layered.

In fact, note that the union of symmetry classes of layered patterns consists of exactly the permutations avoiding patterns in the symmetry classes of $1342, 1423, 2413$.

In [14], Stromquist proved a theorem (later generalized in [1]) on packing layered patterns into permutations.

The inductive proof of this theorem defines a permutation (or a poset) π to be *layered on top* (or *LOT*) if any of its maximal elements is greater than any non-maximal element. The set of these maximal elements is called the *final layer* of π (even if π is not necessarily layered).

Proposition 1.7. *Let* Π *be a multiset of LOT permutations (not necessarily all distinct or of the same length). Then there is an LOT permutation* σ^* *which maximizes the expression*

$$\nu(\Pi, \sigma) = \sum_{\pi \in \Pi} a_\pi \nu(\pi, \sigma), \quad a_\pi \geq 0. \tag{1.1}$$

Furthermore, if the final layer of every $\pi \in \Pi$ *has size greater than 1, then every such* σ^* *is LOT.*

Applying this proposition inductively, [1], following [14], obtains

Theorem 1.8. *Let* Π *be a multiset of layered permutations. Then there is a layered permutation* σ^* *which maximizes the expression (1.1). Furthermore, if all the layers of every* $\pi \in \Pi$ *have size greater than 1, then every such* σ^* *is layered.*

Following [1, 12], we will also define the *ℓ-layer packing density* $\delta_\ell(\Pi)$ for sets of layered permutations Π as the packing density of Π among the permutations with at most ℓ layers. It was shown in both of the above papers that $\delta(\Pi) = \lim_{\ell \to \infty} \delta_\ell(\Pi)$.

2 Monotone patterns

The easiest type of patterns with repeated letters are those whose letters are non-decreasing (or non-increasing) from left to right. By analogy with layered patterns, we will consider nondecreasing patterns.

Theorem 2.1. *Let* $\Pi \in [l]^m$ *be a set of nondecreasing patterns* π *and let* $a_i(\pi)$ *be the number of* i*'s in* π*. For each* $\pi \in \Pi \subseteq [l]^m$*, let* $\hat{\pi} \in S_m$ *be the layered pattern with layer lengths* $(a_1(\pi), \ldots, a_l(\pi))$*, and let* $\hat{\Pi} = \{\hat{\pi} \mid \pi \in \Pi\}$*. Then* $\delta(\Pi, k) = \delta_k(\hat{\Pi})$ *and* $\delta'(\Pi) = \delta(\Pi) = \delta(\hat{\Pi})$*.*

Proof. There is an natural bijection between nondecreasing patterns on k letters and layered patterns with k layers. If π is a nondecreasing pattern with layer lengths $(a_1(\pi), \ldots, a_l(\pi))$, then the map f of Theorem 1.4 induced by the map $i_j \mapsto \sum_{r=1}^{i} a_r(\pi) - j + 1$ (where i_j is the jth i from left) maps π to $\hat{\pi} \in S_m$. Clearly, f^{-1} is induced by a map with takes each element in the ith layer (the ith basic subsequence, in general) to integer i. □

Example 2.2. *Using the results of Price [12], we obtain* $\delta(112) = \delta(\widehat{2}\widehat{13}) = 2\sqrt{3} - 3$*,* $\delta(1122) = \delta(\widehat{21}\widehat{43}) = 3/8$*. More generally, for* $k \geq 2$*,*

$$\delta(\underbrace{1\ldots1}_{k}2) = ka(1-a)^{k-1}, \quad \text{where} \ \ 0 < a < 1, \ \ ka^{k+1} - (k+1)a + 1 = 0.$$

Similarly, for $r, s \geq 2$*,*

$$\delta(\underbrace{1\ldots1}_{r}\underbrace{2\ldots2}_{s}) = \delta(\underbrace{1\ldots1}_{r}\underbrace{2\ldots2}_{s}, 2) = \binom{r+s}{r,s}\frac{r^r s^s}{(r+s)^{r+s}}.$$

Using the results of Albert et al. [1], we also find that $\delta(1123) = \delta(1233) = \delta(1243) = 3/8$*,* $\delta(\{122, 112\}) = \delta(\{132, 213\}) = 3/4$*.*

3 Weakly layered patterns

Again, by analogy with layered permutations, we define weakly layered strings as follows.

Definition 3.1. *A string* $\pi \in [l]^m$ *is* weakly layered *if it is a concatenation of a nondecreasing sequence of non-increasing substrings. In other words,* $\pi = \pi_1 \ldots \pi_r$*, where* π_i *are non-increasing, and* $\pi_1 \leq \cdots \leq \pi_r$ *(that is any letter of* π_i *is not greater than any letter of* π_j *if* $i \leq j$*). Substrings* π_i *maximal with respect to these properties are called the layers of* π*.*

It follows that the consecutive layers of a weakly layered pattern may have at most one letter value in common, for example, $\widehat{1}\widehat{21}$, $\widehat{21}\widehat{2}$, $\widehat{1}\widehat{321}$, $\widehat{1}\widehat{2}\widehat{32}$, $\widehat{21}\widehat{32}$, $\widehat{22111}\widehat{332}$. However, 1231 is not weakly layered.

Theorem 3.2. *If* Π *is a set of weakly layered patterns none of which contains a layer of length 1, then for each* n *and* k*, all maximal* Π*-containing strings in* $[k]^n$ *are weakly layered.*

Proof. If f is an operation as in Theorems 1.4 and 2.1 and π is weakly layered, then $f(\pi)$ is layered. Let $f(\Pi) = \{f(\pi) \mid \pi \in \Pi\}$. It is easy to see that if σ is a maximal Π-containing string, then $f(\sigma)$ is a maximal $f(\Pi)$-containing string. If such σ is non-weakly layered, then it contains a pattern 231 or a pattern 312, hence, 231 or 312 also occurs in $f(\sigma)$, so $f(\sigma)$ is non-layered. But by Theorem 2.2 of [1], $f(\sigma)$ must be layered, contradicting our assumption. Thus, every maximal Π-containing string σ is weakly layered. □

Conjecture 3.3. *If Π is a set of weakly layered patterns, then $\delta'(\Pi) = \delta(\Pi)$ and among maximal Π-containing strings in $[k]^n$, there is one which is weakly layered.*

Note that some maximal Π-containing strings in the above conjecture may not be weakly layered. For example, 12121 is a maximal 121-containing string in $[2]^5$.

We will now find the packing density of some specific weakly layered patterns.

Theorem 3.4. $\delta(121) = \sqrt{3} - 3/2 = \frac{1}{2}\delta(112) = \frac{1}{2}\delta(213)$.

Proof. We will begin with the pattern $\pi = 121$. Let $\sigma = \sigma(n, k)$ be a maximal 121-containing string in $[k]^n$. Without loss of generality, we can assume the smallest letter of $\sigma(n, k)$ is 1, next smallest letter is 2, etc. It is easy to see that σ should begin and end with 1.

Let σ contain n_1 1's. Let $a > 1$ be a letter in σ and m_a and b_a be the numbers of 1's to the left and to the right of a, respectively. Then $m_a + b_a = n_1$, and a occurs in $m_a b_a \le \left\lfloor \frac{n_1^2}{4} \right\rfloor$ patterns 121 in σ which involve the letter 1. The equality certainly occurs for each a when all the 1's of σ are at the beginning or at the end of σ. Consequently, $\sigma = 1 \dots 1\sigma_2 1 \dots 1$, where σ_2 is a string on letters 2 and greater, is maximal 121-containing. Note that σ_2 is also maximal 121-containing.

Following Price [12], we will find the asymptotic ratio $\alpha = \lim_{n\to\infty} \frac{n_1}{n}$. Then it is easy to see that if n_r is the number of letters r in σ, we must have $\lim_{n\to\infty} \frac{n_r}{n} = \alpha(1-\alpha)^{r-1}$.

Since all the 1's of σ are at the beginning or at the end of σ, it is easy to see that half of them should be in the initial block of 1's and the other half, in the terminal block of 1's. Therefore, we have

$$d(121, \sigma) = \max_{0 \le n_1 \le n} \left(d(121, \sigma_2) + \left\lfloor \frac{n_1^2}{4} \right\rfloor (n - n_1) \right)$$

Now the same calculations as in [12, Theorem 5.2] yield

$$\delta(121) = \frac{3}{2} \max_{\alpha \in [0,1]} \frac{\alpha^2(1-\alpha)}{1-(1-\alpha)^3},$$

so $\alpha = (3-\sqrt{3})/2$, $1-\alpha = (\sqrt{3}-1)/2$, and $\delta(121) = \sqrt{3} - 3/2$. $\square$

Here is the complete inventory of packing densities of 3-letter patterns by symmetry class.

Symmetry class	111	112	121	132	123
Packing density	1	$2\sqrt{3}-3$	$\frac{2\sqrt{3}-3}{2}$	$2\sqrt{3}-3$	1

4 Generalized patterns

Generalized patterns were introduced by Babson and Steingrímsson [2] and allow the requirement that some adjacent letters in a pattern be adjacent in its occurrences in an ambient string as well. For example, an occurrence of a generalized pattern 21-3 in a permutation $\pi = a_1 a_2 \cdots a_n$ is a subsequence $a_i a_{i+1} a_j$ of π such that $a_{i+1} < a_i < a_j$. Clearly, in the new notation, classical patterns are those with all hyphens, such as 1-3-2.

Notation 4.1. *This notation (introduced in [2]) may be a little confusing since classical patterns (the ones with all hyphens) were previously written the same way as the generalized patterns with all adjacent letters (i.e. with no hyphens). From now on, we will use the generalized pattern notation. However, if we consider subword patterns (those with no hyphens), we may write π_g for a generalized pattern π without hyphens where the context allows for ambiguity.*

As with the classical patterns, considered in the earlier sections, most papers on generalized patterns deal with pattern avoidance. For example, Claesson [8] and Claesson and Mansour [9] considered the number of permutations avoiding one or two generalized patterns with one hyphen. Burstein and Mansour [6] looked at the same problem with repeated letters allowed in both in the pattern and the ambient string. Elizalde and Noy [10] and Burstein and Mansour [7] considered generalized patterns without hyphens, i.e. with all consecutive letters adjacent.

Here we consider packing generalized patterns into words.

If $\pi \in [l]^m$ is a generalized pattern with b blocks of consecutive letters (i.e. $b-1$ hyphens), then it is easy to see by considering the positions of the first letters of the blocks of π that the maximum possible number of times π can occur in $\sigma \in [k]^n$ is

$$\binom{n-m+b}{b} \sim \frac{n^b}{b!} \quad \text{as } n \to \infty$$

(this yields $\binom{n}{m}$ when $b = m$, i.e. when π is a classical pattern).

In fact, this maximum is achieved when π is a *constant* generalized pattern, i.e. any of the generalized patterns obtained from the constant strings $11\ldots1$ by inserting hyphens at arbitrary positions (possibly, none). Obviously, maximal π-containing strings are the constant strings of length n. Thus, any set of constant generalized patterns has packing density 1. Similarly, any set Π of hyphenated identity generalized patterns has $\delta(\Pi) = 1$.

Given a set of generalized patterns with b blocks, $\Pi \subseteq [l]^m$, we define the packing density of Π similarly to that of a set of classical patterns. We will use the same notation as in Section 1 for the generalized patterns.

It is not hard to see that the analog of Theorem 1.4 holds for generalized patterns as well.

Theorem 4.2. *Let $\Pi \subseteq S_m$ be a set of generalized permutation patterns, then the packing density of Π on words is equal to that on permutations.*

Proof. The same argument as in Theorem 1.4 shows that among maximal Π-containing strings in $[n]^n$ there is one that has no repeated letters. □

4.1 Generalized patterns without hyphens

The maximal number of occurrences of a generalized pattern in $[l]^m$ without hyphens (i.e. with $b = 1$ blocks) is $\binom{n-m+1}{1} = n - m + 1 \sim n$ as $n \to \infty$.

Theorem 4.3. *Let $\pi \in [l]^m$ be a nonconstant, nonidentity monotone generalized pattern without hyphens in which each letter i occurs m_i times. Let $M_\pi = \max(m_1, \ldots, m_l)$. Then $\delta(\pi) = \delta'(\pi) = 1/M_\pi$.*

Proof. Let $\sigma \in [k]^n$ be a word with maximal π-containing word, then it is easy to see that σ has the form $\sigma = \sigma'\sigma' \cdots \sigma'\sigma''$, where $\sigma' = 11\ldots122\ldots2\ldots(k-$

$1)(k-1)\dots(k-1)k\dots k$ such that every letter $1,2,\dots\dots,k-1$ appears M_π times, k appears m_k times, and σ'' is a prefix of σ'. Hence, if $n' = n - \text{length}(\sigma'')$ (so $n - M_\pi(k-1) - m_k < n' \le n$), then

$$\frac{\frac{n'}{M_\pi(k-1)+m_k}(k-1)}{n-m+1} \le \frac{\mu(\pi,n,k)}{n-m+1} \le \frac{\frac{n-m_k}{M_\pi(k-1)+m_k}(k-1)}{n-m+1}.$$

Therefore, $\delta(\pi) = \delta'(\pi) = 1/M_\pi$. □

Theorem 4.4. *Let $\pi = (\phi_1,\dots,\phi_s) \in [l]^m$ be any s-layered generalized pattern without hyphens such that $s > 1$. Let $M_\pi = \max_{1\le j\le s}|\phi_j|$. Then $\delta(\pi) = \delta'(\pi) = 1/M_\pi$.*

Proof. The same mapping as in Theorem 2.1 shows that our π has the same packing density as the corresponding monotone generalized pattern without hyphens of Theorem 4.3. □

Corollary 4.5. *Let $\pi_1 = 11\dots12_g \in [2]^m$ and $\pi_2 = 1m(m-1)\dots2_g \in [m]^m$, then $\delta(\pi_1) = \delta'(\pi_1) = 1/(m-1)$ and $\delta(\pi_2) = \delta'(\pi_2) = 1/(m-1)$.*

For instance, $\delta(112_{\mathrm{g}}) = \delta'(112_{\mathrm{g}}) = 1/2$, $\delta(132_{\mathrm{g}}) = \delta'(132_{\mathrm{g}}) = 1/2$, $\delta(123_{\mathrm{g}}) = \delta'(123_{\mathrm{g}}) = 1$.

4.2 Generalized patterns with one hyphen

The maximal number of occurrences of a generalized pattern in $[l]^m$ with one hyphen (i.e. with $b = 2$ blocks) is $\binom{n-m+2}{2} \sim n^2/2$ as $n \to \infty$.

Proposition 4.6. $\delta(11\text{-}2) = \delta'(11\text{-}2) = 1$.

Proof. Let $\sigma \in [k]^n$ be a maximal (11-2)-containing word, then σ is a monotone nondecreasing string in which letter i occurs n_i times, $n_1 + \cdots + n_k = n$. Then $\mu(11\text{-}2,n,k) = \max\{\sum_{i=1}^k (n_i-1)(n_{i+1}+\cdots+n_k) \ : \ n_1+\cdots+n_k = n\}$. From here, it is not difficult to determine that $\mu(11\text{-}2,n,k) \sim n^2/2$ as $n \to \infty$. Choose n_i's to be such integers that $|n_i - \frac{n}{k}| < 1$ and $|n_1 + \cdots + n_r - \frac{rn}{k}| < 1$ for each $r = 1,2,\dots,k$. Then

$$\mu(11\text{-}2,n,k) \sim \left(\frac{n}{k}\right)^2 \binom{k}{2},$$

out of $\binom{n-1}{2}$ maximum possible occurrences, and the result follows. □

Proposition 4.7. $\delta(12\text{-}1) = \delta'(12\text{-}1) = 1/3$.

Proof. Let $\sigma \in [k]^n$ be a word with maximum occurrences of 12-1, then

$$\sigma = 1212\cdots1211..1 \in [2]^n$$

where the string 12 occurs in α exactly d times. So $\mu(12\text{-}1,n,k) = \max_{1\le d\le n}(d(d-1)/2 + d(n-2d))$, and the maximum occurs at $d \sim n/3$. The rest is easy to check. □

Proposition 4.8. $\delta(12\text{-}3) = \delta(21\text{-}3) = 1$.

Proof. For pattern 12-3, consider the identity permutation. For pattern 21-3, consider the layered permutation with layers of equal length. □

References

[1] M.H. Albert, M.D. Atkinson, C.C. Handley, D.A. Holton, W. Stromquist, On packing densities of permutations, *Electronic J. of Comb.* **9** (2002), #R5.

[2] E. Babson, E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, *Séminaire Lotharingien de Combinatoire*, B44b:18pp, 2000.

[3] M. Bóna, B.E. Sagan, V.R. Vatter, Pattern frequency sequences and internal zeros, *Adv. in Appl. Math.* **28** (2002), 395-420.

[4] A. Burstein, Enumeration of words with forbidden patterns, Ph.D. thesis, University of Pennsylvania, 1998.

[5] A. Burstein, T. Mansour, Words restricted by patterns with at most 2 distinct letters, *Electronic J. of Combinatorics* **9(2)** (2002), #R3.

[6] A. Burstein, T. Mansour, Words restricted by 3-letter generalized multipermutation patterns, preprint. Available at `www.arXiv.org/ps/math.CO/0112281`.

[7] A. Burstein, T. Mansour, Counting occurrences of some subword patterns, preprint. Available at `www.arXiv.org/ps/math.CO/0204320`.

[8] A. Claesson, Generalized pattern avoidance, *European J. of Combinatorics* **22:7** (2001), 961–971.

[9] A. Claesson, T. Mansour, Counting occurrences of a pattern of type $(1, 2)$ or $(2, 1)$ in permutations, *Adv. in Appl. Math.*, **29** (2002), 293–310.

[10] S. Elizalde, M. Noy, Enumeration of subwords in permutations, Local Proceedings of *Formal Power Series and Algebraic Combinatorics (FPSAC'01), 13th International Conference*, ed. by H. Barcelo and V. Welker, Tempe, Arizona (USA), May 20-26, 2001, 179–189.

[11] D.E. Knuth, The Art of Computer Programming, vols. 1, 3, Addison-Wesley, NY, 1968, 1973.

[12] A. Price, Packing densities of layered patterns, Ph.D. thesis, University of Pennsylvania, 1997.

[13] R. Simion, F.W. Schmidt, Restricted Permutations, *Europ. J. of Comb.* **6** (1985), 383–406.

[14] W. Stromquist, Packing layered posets into posets, manuscript, unpublished, 1993.

[15] R. Tarjan, Sorting using networks of queues and stacks, *J. of the Association for Computing Machinery* **19** (1972), 341–346.

Some Contributions to the Coupon Collector Problem

Anant P. Godbole

Consider a sequence $\{\pi_1, \pi_2, \ldots\}$ of random permutations of $\{1, 2, \ldots, n\}$ and let

$$N = \inf\{m : \{\pi_1(1), \pi_2(1), \ldots, \pi_m(1)\} = \{1, 2, \ldots, n\}\}.$$

The random variable N, rephrased as above in the language of permutation patterns, represents the key variable of interest in the so-called "coupon collector" problem, where the object is to collect one of each of n different types of "toys" that are randomly distributed in "cereal boxes". How many cereal boxes must one buy to achieve one's goal? Several approximations to the distribution of N may be found in the literature; these include normal; Poisson-based; log-normal; and saddle point approximations. We will first argue that the normal approximations we present yield excellent results when n grows very large, and that claims of superiority of other distributions are true only for small to moderate values of n.

An auxiliary variable of interest is X_r, which is the number of times, at the completion of the coupon collection process, that the rth toy to appear is collected. It is clear that $X_n = 1$, and that there is a form of stochastic monotonicity in the variables $\{X_r : r \geq 1\}$. We use the Stein-Chen method (see, e.g., Barbour, A., Holst, L., and Janson, S. (1992). *Poisson Approximation*, Oxford University Press.) to show that good univariate and multivariate Poisson approximations may respectively be obtained for the variables X_r, and for the ensemble $\{X_s : 1 \leq s \leq t\}$, where t is not too large.

Finally, we address a question brought to our attention at the 2001 "Random Structures and Algorithms" conference (held in Poznán, Poland) by Doron Zeilberger, and researched recently by persons such as Foata, Myers, Wilf and Zeilberger: Let W_r be the number of toys that appear exactly r times at the end of the collection process. Clearly $W_1 \geq 1$, and, in fact, the distribution of W_1 can be approximated quite well, but the situation gets extremely delicate for values of $r \geq 2$. We present some results along these lines.

This is joint work with Alina Badus (Carleton College), Natalie Lents (Centre College) and Erin LeDell (Trinity College).

The packing density of layered permutations

Peter A. Hästö

The purpose of this talk is to present some results on the packing density of layered permutations. As an introduction the work of W. Stromquist [4], A. Price [3] and M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton & W. Stromquist [1] is considered. The main part of the talk consists of presenting some extensions of mine of these results, from [2].

Let $\sigma \in S_n$ (the symmetric group of n letters) and $\pi \in S_m$. The number of occurrences of π in σ is the number of m element subsets E of $[n] := \{1, 2, \dots, n\}$ such that $\sigma|_E$ and π are isomorphic (as mappings of ordered sets). For instance the permutation 1374625 contains 5 occurrences of the permutation 1423, namely 1746, 1745, 1725, 3746 and 3745. The packing density problem consists of finding the maximum number of containments of a given permutation.

Let us denote the number of times that $\pi \in S_m$ is contained in $\sigma \in S_n$ by $\nu(\pi, \sigma)$. If we divide this number by the total number of subsequences of σ of length m (for $m \le n$) we get the density of π in σ:

$$d(\pi, \sigma) := \frac{\nu(\pi, \sigma)}{\binom{n}{m}}.$$

Since we want to determine the maximum number of containments, we further define

$$d_n(\pi) := \max_{\sigma \in S_n} d(\pi, \sigma).$$

We say that a permutation $\sigma \in S_n$ is π-*maximal* if $d_n(\pi) = d(\pi, \sigma)$. It turns out that $d_n(\pi)$ is decreasing in n and hence it makes sense to define the *packing density of* π by

$$d(\pi) := \lim_{n \to \infty} d_n(\pi)$$

(this is proved in [1, Proposition 1.1], although the authors of that paper consider it a part of combinatorial folklore).

Since the packing density problem seems to be quite difficult in general we restrict our attention to the packing density of layered permutations. We say that the permutation $\pi \in S_m$ is *layered* if there exist numbers $m_1, \dots, m_r$, the sum of which equals m, such that π starts with the m_1 first positive integers in reverse order, followed by the next m_2 positive integers in reverse order and so on. More specifically, we say that this permutation is of type $[m_1, \dots, m_r]$. For instance 213654 is layered of type $[2, 1, 3]$. Notice that the type of a layered permutation uniquely determines the permutation. The nice thing about considering layered permutations is that W. Stromquist [4] proved:

Theorem 2.2, [1]. *Let π be a layered permutation. Among the π-maximal permutations of each length there will be one that is layered. Furthermore, if all the layers of π have size greater than* 1*, then every π-maximal permutation is layered.*

In [2] only the packing density of layered permutations was considered and therefore the following convention was introduced:

Notation. Throughout this abstract we denote by π a layered permutation of type $[m_1, \ldots, m_r]$ and by m the sum $m_1 + \ldots m_r$. All other permutations are also assumed to be layered, unless specified to the contrary.

The central theme of the results in [2] is the number of layers in near π-maximal permutations. It was shown in [3] for some permutations the number of layers in π-maximal $\sigma_n \in S_n$ is bounded as $n \to \infty$ whereas for others it is unbounded (we say that these are of the *bounded* and *unbounded type*, respectively). Albert, Atkinson, Handley, Holton & Stromberg (hereafter referred to as AAHH&S) stated the following conjecture.

Conjecture 2.9, [1]. *Suppose that π is a layered permutation whose first and last layers have size greater than* 1 *and which has no adjacent layers of size 1. Then π is of the bounded type.*

These authors showed that the conjecture is true when we consider only layered permutations with at most three layers [1, Proposition 2.8] or permutations with every layer of size two or greater [1, Theorem 2.7]. Also, in Proposition 2.10 they showed that the assumption on the first and last layers is necessary. Knowing that a permutation is of the bounded type has certain implications, in particular it allows us to estimate (and in principle, also to calculate the exact value of) the packing density by finding a maximum of a certain function introduced by Price in [3]. Nevertheless, the bounds on the number of layer given by the previous finiteness results are so large that they are virtually useless in determining the packing density. For instance, Theorem 2.7 of [1] implies that the number of layers in a π-maximal permutation for π of type $[2, 3, 2]$ is less than 30 [1, p. 19] whereas AAHH&S suggested that the correct number in this case should be three. The contribution of [2] are two results which apply only to more limited classes of layered permutations, but conversely give optimal bounds for the number of layers in near π-maximal permutations.

Let us say that the layered permutation π is *simple* if there exists a sequence $\{\sigma_n\}$ with $\sigma_n \in S_n$ such that every σ_n has r layers and $\lim_{n\to\infty} d(\pi, \sigma_n) = d(\pi)$. It turns out that it is very easy to calculate the packing density of a simple permutation, see [3, Theorem 4.1]. The next result shows that there are many simple permutations:

Theorem 1.2, [2]. *Let $\pi \in S_m$ be a layered permutation of type $[m_1, \ldots, m_r]$. If $\log_2(r + 1) \le \min\{m_i\}$ then π is simple and*

$$d(\pi) = \frac{m!}{m^m} \prod_{k=1}^{r} \frac{m_k^{m_k}}{m_k!},$$

where $m := m_1 + \ldots + m_r$.

In [2, Lemma 3.5] it is shown that there exists a permutation with

$$\min\{m_i\} \le \frac{\log(r + 1)}{r \log(1 + 1/r)}$$

which is not simple. This implies that the logarithmic bound in the previous theorem is asymptotically off by at most a factor of $1/\log 2$.

Notice that the previous theorem solves the packing density problem for layered permutations with two or three layers none of which is a *singleton* (i.e. has length 1). Since A. Price [3] has previously solved the packing density problem for permutations of the type $[1, k]$ this means that the we now know how to handle all the two layer cases.

The (non-trivial) layered permutations with three layers not covered by the theorem are of type $[1, k_1, k_2]$, $[1, k_1, 1]$, $[1, 1, k_1]$ or $[k_1, 1, k_2]$ (with $k_1, k_2 \ge 2$). Recall that

k	Upper	Lower	Rel. error
3	0.124502	0.133108	$6.912 \cdot 10^{-2}$
4	0.106597	0.107403	$7.567 \cdot 10^{-3}$
5	0.094881	0.094941	$6.300 \cdot 10^{-4}$
6	0.086331	0.086335	$4.253 \cdot 10^{-5}$

Table 1: Estimates of the packing density of the permutation of type $[k, 1, k]$

the first three of these were shown to be of the unbounded type in Proposition 2.10, [1], which suggests that it will be difficult to calculate or estimate their packing density (it might be possible to handle the case $[1, 1, k_1]$ as in [1, Proposition 2.4] but the generalization is not straightforward). Section 4 of [2] is devoted to a special case of the fourth type, $[k_1, 1, k_2]$.

It turns out that permutations with a singleton layer are never simple, however, in some cases near π-maximal permutation can be chosen to have exactly one layer more than the packed permutation. More precisely, let us say that a permutation π is almost simple if it is not simple, but there there exists a sequence $\{\sigma_n\}$ with $\sigma_n \in S_n$ such that every σ_n has $r + 1$ layers and $\lim_{n\to\infty} d(\pi, \sigma_n) = d(\pi)$.

Theorem 1.3, [2]. *Let π be a layered permutation of type $[k, 1, k]$ with $k \geq 3$. Then π is almost simple.*

This result gives us very good estimates of the packing densities of these permutationsm, see Table 1. Unfortunately, the case $[2, 1, 2]$ is not covered, which means that we are not able to answer the question asked in [1, p. 19] regarding the packing density of this permutation.

The talk is concluded by some remarks on why it seems to be difficult to use the methods of [2] to tackle the conjecture of AAHH&S on which permutations are of the bounded type.

References

[1] M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton & W. Stromquist, On packing densities of permutations, *Electron. J. Combin.* **9** (2002), #R5.

[2] P. A. Hästö, The packing densities of other layered permutations, *Electron. J. Combin.* **9(2)** (2002), #R1.

[3] A. Price, Packing densities of layered patterns, Ph. D. thesis, University of Pennsylvania, Philadelphia, PA, 1997.

[4] W. Stromquist, Packing layered posets into posets, Unpublished typescript, 1993.

Permutation polytopes

Frank K. Hwang and Uriel G. Rothblum

Let p be a positive integer. A real-valued function λ over subsets of $\{1, 2, \dots, p\}$ defines a mapping of permutations of $\{1, 2, \dots, p\}$ into p-vectors where coordinate i of the vector corresponding to a permutation is the augmented λ-value obtained from adding i to the coordinates that precede it. The permutation polytope corresponding to λ is then the convex hull of the vectors corresponding to all permutations. When λ is supermodular, we derive a characterizing system of linear inequalities for the permutation polytope, we show the the "spanning vectors" are the vertices of the polytope and show the the directions of the edges are differences of unit vectors. Further, under tightened supermodularity conditions, isomorphic representation of the face lattice are obtained. Applications to partitioning problems and to convex games are discussed.

Prefix Exchanging and Pattern Avoidance by Involutions

Aaron D. Jaggard[*]
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395 USA
adj@math.upenn.edu

Abstract

Let $I_n(\pi)$ denote the number of involutions in the symmetric group $\mathcal{S}_n$ which avoid the permutation π. We say that two permutations $\alpha, \beta \in \mathcal{S}_j$ *may be exchanged* if for every k and ordering τ of $j+1, \ldots, k$, we have $I_n(\alpha\tau) = I_n(\beta\tau)$ for every n.

Our primary results are that both the prefixes 12 and 21 and the prefixes 123 and 321 may be exchanged. The first result gives a number of known results for patterns of length 4 and some new results for longer patterns. The second implies a conjecture of Guibert, thus completing the classification of $\mathcal{S}_4$ with respect to pattern avoidance by involutions, and gives additional results for longer patterns. Our approach parallels that used by Babson and West to prove analogous results for pattern avoidance by general permutations (without the restriction to involutions), with some modifications made necessary by the symmetry of the current problem.

1 Introduction and Results

The *pattern* of a sequence $w_1w_2\ldots w_k$ of k distinct letters is the order preserving relabelling of the sequence with $[k] = \{1, 2, \ldots, k\}$. Given a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ in the symmetric group $\mathcal{S}_n$, we say that π *avoids* the pattern $\sigma = \sigma_1\sigma_2\ldots\sigma_k \in \mathcal{S}_k$ there is no subsequence $\pi_{i_1}\ldots\pi_{i_k}$, $i_1 < \cdots < i_k$, whose pattern is σ.

Let $I_n(\sigma)$ denote the number of involutions (permutations whose square is the identity permutation) in $\mathcal{S}_n$ which avoid the pattern σ, and write $\sigma \sim \sigma'$ if for every n, $I_n(\sigma) = I_n(\sigma')$ (we also say that σ and σ' are in the same *cardinality class*). For $\alpha, \beta \in \mathcal{S}_j$, we say that the prefixes α and β *may be exchanged* if for every $k \geq j$ and ordering $\tau = \tau_1\tau_2\ldots\tau_{k-j}$ of $[k] \setminus [j]$ we have $\alpha\tau \sim \beta\tau$ (*i.e.*, if the patterns $\alpha_1\ldots\alpha_j\tau_1\ldots\tau_{k-j}$ and $\beta_1\ldots\beta_j\tau_1\ldots\tau_{k-j}$ are $\sim$-equivalent for every possible choice of τ).

Our primary results are the following two theorems and their corollaries.

Theorem 3.2. *The prefixes* 12 *and* 21 *may be exchanged.*

This implies some of the known $\sim$-equivalences for patterns in $\mathcal{S}_4$, the most notable being $1234 \sim 2143$, as well as some new results for longer patterns.

[*]Partially supported by the DoD University Research Initiative (URI) program administered by the Office of Naval Research under Grant N00014-01-1-0795, and by NSF Grant CCR-0098096.

Theorem 4.2. *The prefixes* 123 *and* 321 *may be exchanged.*

This implies $1234 \sim 3214$, which was conjectured by Guibert ([2], as reported in [3, 4]). The resolution of this conjecture completes the classification of $\mathcal{S}_4$ according to $\sim$-equivalence. This second theorem implies additional new results for longer patterns.

Both of these theorems follow from Theorem 2.4, stated below, as applied to appropriate lemmas. This parallels the approach of Babson and West [1], who proved the results analogous to Theorems 3.2 and 4.2 for pattern avoidance by general permutations (without the restriction to involutions).

Section 2 gives some preliminary definitions and uses these to state our general theorem giving sufficient conditions for being able to exchange prefixes. Theorem 3.2, related lemmas, and some of its corollaries are stated in Section 3, while Theorem 4.2, related lemmas, and some corollaries are stated in Section 4.

2 Preliminaries and General Results

We start by placing dots in the boxes of partition diagrams, which we take to have their longest row at the bottom and which we coordinatize from the bottom left corner.

Definition 2.1. Given a partition λ, a *placement on* λ is an assignment of dots to some of the boxes in the Young diagram of λ such that no row or column contains more than one dot. These are often thought of as non-attacking rooks on a generalized chessboard of shape λ. We call the placement *full* if each row and column of λ contains exactly 1 dot. We define the *transpose* of a placement to be the placement which has a dot in box (i, j) iff the original placement had a dot in box (j, i). The transpose of a placement on λ is a placement on the conjugate λ' of λ. We call a placement on a partition λ *symmetric* if the transpose of the placement is the original placement.

The connection between permutations and rook theoretic language of placements on boards comes through the graphs of permutations.

Definition 2.2. The *graph* of an n-permutation π is the full placement on the $n \times n$ square SQ_n which has dots in exactly the boxes $\{(i, \pi(i))\}_{i \in [n]}$. Note that the graph of π^{-1} is the transpose of the graph of π.

We also define a notion of pattern containment for placements on general boards.

Definition 2.3. Given $\sigma = \sigma_1 \dots \sigma_j \in \mathcal{S}_j$ and a placement on a partition λ, we say that this placement *contains* the pattern σ if there are dots $(x_1, y_1), \dots, (x_j, y_j)$ in the placement with $x_1 < \dots < x_j$ and the pattern of $y_1 \dots y_j$ equal to σ *and* such that if $y_{max} = \max\{y_i\}$, the box (x_j, y_{max}) is contained in the partition λ. If a placement does not contain σ, then that placement *avoids* σ.

Note that an n-permutation π contains the pattern $\sigma \in \mathcal{S}_j$ iff the graph of π, viewed as a placement on SQ_n, contains σ in the sense of Definition 2.3. A general sufficient condition which allows to involutions to be exchanged as prefixes is given by the following theorem.

Theorem 2.4. *Let* $\lambda_{sym}(\sigma)$ *be the number of symmetric full placements on the partition* λ *which avoid the pattern* σ*. Let* α *and* β *be involutions in* $\mathcal{S}_j$*. If, for* *every self-conjugate partition λ we have $\lambda_{sym}(\alpha) = \lambda_{sym}(\beta)$, then the prefixes α and β may be exchanged.*

In order to prove Theorem 2.4, we define a self-conjugate partition for an n-involution π and arrangement τ of $[k] \setminus [j]$.

Definition 2.5. Let π be an involution in $\mathcal{S}_n$ and τ some ordering of $[k] \setminus [j]$ for some $k \geq j$. Take all boxes (x, y) such that the graph of π contains a set of dots $(x_{j+1}, y_{j+1}), \ldots, (x_k, y_k)$ with $x_{j+1} < \cdots < x_k$, where the pattern of the word $y_{j+1}, \ldots, y_k$ is the same as the pattern of τ, and either $x < x_{j+1}$ and $y < \min\{y_i\}$ or $x < \min\{y_i\}$ and $y < x_{j+1}$. This forms a self-conjugate partition on which there is a (not necessarily full) symmetric placement obtained by restricting the graph of π to this partition. Delete the rows and columns of this partition which do not contain a dot to obtain another self-conjugate partition; we call this the *self-conjugate τ-partition of* π and denote it by $\lambda_\tau(\pi)$. The deletion of empty rows and columns yields a full symmetric placement on $\lambda_\tau(\pi)$; we call this the *placement on $\lambda_\tau(\pi)$ induced by π.*

We use these definitions to state and prove the following lemma.

Lemma 2.6. *If π is an n-involution, σ a j-involution, and τ any arrangement of $[k] \setminus [j]$, then π contains the pattern $\sigma\tau \in \mathcal{S}_k$ if and only if $\lambda_\tau(\pi)$ is nonempty and the placement on $\lambda_\tau(\pi)$ induced by π contains σ in the sense of Definition 2.3.*

Proof. (Sketch) If π contains $\sigma\tau$, this is straightforward by the construction of $\lambda_\tau(\pi)$.

If $\lambda_\tau(\pi)$ contains an occurrence of σ, then the graph of π must contain dots whose pattern is that of τ and which are northeast of a copy of either σ or σ^{-1}. By our assumption that σ is an involution, each of these possibilities gives an occurrence of the pattern $\sigma\tau$ in π. $\square$

Proof. (Of Theorem 2.4, sketch) When constructing $\lambda_\tau(\pi)$, note which squares from the graph of π correspond to those in $\lambda_\tau(\pi)$. Consider the set of involutions π for which $\lambda_\tau(\pi) = \mu$ for some fixed μ, which have the same squares in their graphs corresponding to their symmetric τ-shapes, and which agree everywhere outside these squares. Since $\lambda_{sym}(\alpha) = \lambda_{sym}(\beta)$, Lemma 2.6 implies that the number of involutions in this set which avoid $\alpha\tau$ equals the number which avoid $\beta\tau$. Summing over all such sets of involutions completes the proof. $\square$

3 Exchanging 12 and 21

Lemma 3.1. *For any self-conjugate partition λ, the number of symmetric full placements on λ which avoid 12 equals the number which avoid 21.*

Proof. If λ has any full placements, there are unique full placements on λ which avoid 12 and 21 as shown in [1]. If λ is self-conjugate, the reflection of any placement on λ across the diagonal of symmetry gives another placement on λ. This placement avoids 12 (21, respectively) iff the original placement did. By the uniqueness of the full placements which avoid 12 and 21, the reflected placement must coincide with the original one and is thus symmetric. $\square$

Applying Theorem 2.4 to Lemma 3.1 allows us to exchange 12 and 21.

Theorem 3.2. *The prefixes* 12 *and* 21 *may be exchanged.* □

As a corollary, we have many of the previously known equivalences for patterns of length 4. In particular, $1234 \sim 2143$ as conjectured by Guibert [2] and proved more recently by Guibert, Pergola, and Pinzani [4].

Corollary 3.3.

$$1234 \sim 2134 \sim 2143$$

Proof. Symmetry relations give $2134 \sim 1243$, then apply Theorem 3.2. □

Numerical results both suggest possible $\sim$-equivalences and indicate which symmetry classes (*i.e.*, permutations which are trivially $\sim$-equivalent based on symmetry arguments) cannot be in the same cardinality class. Among non-involutions of length 5, there is only one possible $\sim$-equivalence between symmetry classes. Theorem 3.2 shows that this does indeed hold.

Corollary 3.4.

$$12453 \sim 21453$$

Among involutions of length 5, one cardinality class contains at most the symmetry classes 12435 and 21435; these are in fact $\sim$-equivalent.

Corollary 3.5.

$$12435 \sim 21435$$

Numerical results suggest that a large number of symmetry classes may form a single cardinality class with 12345. Theorem 3.2 collapses two of these symmetry classes and a different set of three of these symmetry classes as follows.

Corollary 3.6.

$$12543 \sim 21543$$

Corollary 3.7.

$$12345 \sim 12354 \sim 21354$$

As a corollary of Theorem 4.2, we see that these five symmetry classes are all part of the same cardinality class.

4 Exchanging 123 and 321

We prove that the prefixes 123 and 321 may be exchanged using an approach which parallels that used by Babson and West to prove the analogous result for pattern avoiding permutations (without the restriction to involutions); we symmetrize one of their results as Lemma 4.1. Note that the symmetrized property does not hold for square partitions (consider the graphs of involutions in $\mathcal{S}_3$), which we treat as an additional base case for the induction used in our proof.

Lemma 4.1. *If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a non-square self-conjugate partition then the number of symmetric full placements on λ which avoid* 123 *and have a dot in (i, λ_1), $1 \leq i \leq \lambda_k$, equals the number of symmetric full placements on λ which avoid* 321 *and have a dot in $(\lambda_k + 1 - i, \lambda_1)$.*

We postpone discussion of the proof of Lemma 4.1 and first consider its primary application, the exchanging of the prefixes 123 and 321, and some corollaries of this theorem.

Theorem 4.2. *The prefixes* 123 *and* 321 *may be exchanged.*

Proof. Summing Lemma 4.1 over $1 \le i \le \lambda_k$, the number of 123-avoiding symmetric full placements on a non-square self-conjugate partition equals the number of 321-avoiding such placements. Symmetry of the RSK algorithm gives $123 \sim 321$, so the number of symmetric full placements on SQ_n which avoid 123 equals the number which avoid 321. We may then apply Theorem 2.4. □

This affirmatively answers a conjecture of Guibert (Conjecture 5.3 of [4], originally from [2]; recall that $1432 \sim 3214$ by symmetry), completing the classification of patterns of length 4 according to containment by involutions.

Corollary 4.3.

$$1234 \sim 3214$$

As noted in Section 3, this also has applications for determining the cardinality classes among patterns of length 5. The symmetry classes in Corollaries 3.6 and 3.7 are in the same cardinality class ($12543 \sim 32145$ by symmetry).

Corollary 4.4.

$$12345 \sim 32145$$

In order to prove Lemma 4.6 we start with the following lemma, which follows directly from the symmetry of the RSK algorithm. This gives additional base cases for the symmetrization of the induction used in [1].

Lemma 4.5. *The number of full symmetric placements on SQ_n which avoid the pattern* 123 *and whose leftmost i columns avoid* 12 *equals the number of full symmetric placements on SQ_n which avoid* 321 *and whose rightmost i columns avoid* 21*.*

Proof. We use the language of n-involutions instead of full symmetric placements on SQ_n.

123-avoiding involutions correspond to standard Young tableaux with at most 2 columns. Those which avoid 12 in their first i entries are those whose first i entries form a decreasing subsequence; these correspond to tableaux with at most 2 columns and whose first column contains $1, \dots, i$.

321-avoiding involutions which avoid 21 in their last i entries may be reversed to obtain 123-avoiding permutations which avoid 12 in their first i entries. These correspond to pairs (P, Q) of tableaux which have at most two columns and in which Q contain $1, \dots, i$ in its first column. The pairs of this type which correspond to the reversal of an involution are exactly those in which P is the transpose of the evacuation of the transpose of Q (see Appendix A.1 of [5]). □

Finally, we symmetrize Lemma 2.2 of [1] as follows. This lemma is proved by case analysis (somewhat more extensive than in the asymmetric case); many of the transformations involved resemble those used in the asymmetric case.

Lemma 4.6. *Let λ be a symmetric partition of length k, $i < \lambda_k$, $j \le \lambda_k - i$. The number of symmetric full placements on λ which avoid* 321 *and which avoid* 21 *in columns $i+1, \dots, i+j$ is the number which avoid* 321 *everywhere and* 21 *in columns $i, \dots, i+j-1$.*

Given a nonsqaure self-conjugate partition λ, let $\hat{\lambda}$ be the self-conjugate partition obtained by deleting the leftmost and rightmost columns and top and bottom rows of λ. We prove Theorem 4.1 by an induction on $\hat{\lambda}$ which closely follows that in [1], making use of Lemma 4.6. In the case that $\hat{\lambda}$ is a square, we invoke Lemma 4.5.

References

[1] Eric Babson and Julian West, *The permutations* $123p_4 \cdots p_m$ *and* $321p_4 \cdots p_m$ *are Wilf-equivalent*, Graphs Combin. **16** (2000), no. 4, 373–380.

[2] O. Guibert, *Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de young*, Ph.D. thesis, University Bordeaux 1, 1995.

[3] O. Guibert and T. Mansour, *Restricted* 132*-involutions and Chebyshev polynomials*, arXiv:math.CO/0201136.

[4] O. Guibert, E. Pergola, and R. Pinzani, *Vexillary Involutions are Enumerated by Motzkin Numbers*, Ann. Comb. **5** (2001), no. 2, 153–174.

[5] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge University Press, Cambridge, 1999.

Fair ranking in dancesport competitions

X. Munoz
Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
Spain
`xml@mat.upc.es`

Abstract

The dancesport scrutineering problem is a particular instance of the general problem of combining several particular preferences into a global one. The desired properties for such a scrutineering differ however from other contexts of social welfare functions. In this paper we deal with the Kemeny-Young method, focusing on the properties which are desirable in dancesport. We also give a new interpretation of Kemeny-Young method in terms of graph theory, which allows to modify it in order to avoid some of the possible ties.

1 Introduction

An extensive literature has been devoted to the design of social welfare functions, i.e. rules for aggregating individual preferences or rankings into a global preference or final ranking. probably because of the large number of contexts in which social welfare functions are needed (artificial intelligence, voting schemes, professional sport,...).

Even, the number of different social welfare functions used in professional sport is almost as large as the number of sports. This is not surprising, since different sports have different rules for competition and different ways to obtain particular rankings which should be aggregated to a collective final ranking. Due to these differences, a ranking method may be appropriate for a certain sport and inappropriate or even unfair for another one.

It is not the same, for instance, a soccer league, in which particular rankings are obtained by pairwise competitions, and a car race competition, in which all participants meet in all races. It is also different the case in which only the first ranked participants are to be classified as in the eliminatories on olympic athletics, to the case in which the whole ranking is required, as in the tennis ATP ranking, or to the case in which only an olympic medal is to be obtained. Sometimes all preferences may be considered to have the same weight, as in pairwise competition in a basketball league, and in some other cases, some asymmetry is desirable, as in pairwise chess competitions. Rankings in judge competitions such as synchronized swimming or skating have different requirements that rankings coming from measurable objective data such in length jump, in which manipulability of judges does seldom make sense. Notice that there are even sports in which rankings are obtained both from judges and from objective data, like in sky jump.

But even focussing on judge competitions, not all competitions are equivalent. In some sports, participants compete in sequential order, and they are scored by judges as in gymnastics, and in some others all participants compete together and they are not scored but ranked by judges, as in dancesport.

Since we will focus, in this paper, on dancesport competitions, let us first of all explain the particularities of this sport.

The goal of a dancesport competition is to sort the participant couples according to the quality in the performance of a certain number of dances. This is done by means of a certain number of judges, typically not less than 3 and not larger than 21.

If the number of couples exceeds a certain value, then the competition includes one or more eliminatories in order to do a progressive selection. In any case, the competition ends with a final in which he number of couples is usually less than or equal to 7.

Unequal to other judge sports, all couples dance at the same time and judges do not score them, but only sort them from the best performer to the worst one. Each judge acts independently from the rest according to his/her point of view, with the only restriction that no ties are allowed in his/her verdict. If the competition consists on different dances, this process is repeated for each dance. The goal of the social welfare function is to combine these multiple rankings in order to obtain a global ranking of all couples. Table 1(a) shows a typical verdict of judges for a single dance. In this example there are 7 couples with dorsals 11, . . . , 17, and 5 judges labelled with letters A, . . . , E. Table 1(b) shows the same data, in a different presentation.

Ord	A	B	C	D	E
1	11	11	14	16	12
2	13	12	13	14	13
3	14	14	12	12	16
4	12	15	17	13	15
5	17	16	15	17	14
6	15	13	16	15	17
7	16	17	11	11	11

a) ranking for couples.

	A	B	C	D	E
11	1	1	7	7	7
12	4	2	3	3	1
13	2	6	2	4	2
14	3	3	1	2	5
15	6	4	5	6	4
16	7	5	6	1	3
17	5	7	4	5	6

b) Ordinal for couples.

Table 1: A typical ranking table in dancesport competitions

The method used nowadays to obtain the final ranking is the so called *Skating system* [8]. This method has been used without modifications since 1956, but some paradoxes have been recently observed, and yet there are people who claim that this system should be improved. Although many of these paradoxes appear when competitions consist on more than one dance, we will focus in the case of a single dance competition, since the origin of those paradoxes is often the method used to rank a single dance.

The remaining sections of the paper are structured the following way: In Section 2 we present the desiderable conditions that a ranking method for dancesport as well as some previous results. Section 3 is devoted to give an overview of the *Kemeny-Young* method from the graph theory point of view. In Section 4 the nature of ties in Kemeny-Young method are revisted. Finally in the remanning sections we propose slight modifications in Kemeny-Young method in order to avoid possible ties

2 Setting of the problem and previous results

In order to find a reasonable and fair sorting method it is important to fix the requirements that must hold, but taking into account that the more restrictive the requirements are, the more difficult it will be to find a method that meets all of them. In this sense there is a classical impossibility result from Arrow for social welfare functions [1].

Among the large amount of principles that could be required for the ranking method in dancesport, in order to avoid some of the paradoxes, X.Mora [9], proposed to focus on the following ones:

1. *Majority Principle* (MP). If a couple is ranked in the first place by an absolute majority of the judges, then it should be the winner in the final ranking. This principle is also known as *Condorcet principle*.
2. *Independence of irrelevant alternatives* (IIA). Positions in the ranking should not be altered when removing the worst ranked couple.
3. *Discrimination*. The less the number of ties, the better.

There are other standard requirements that should also hold, although not mentioned in [9], such as *neutrality* (the method should be symmetric in the treatment of all participants), or *consistency* (if two set of judges arrive to the same consensus ordering, then meeting together this should still be their consensus). The reader is referred to [11] for an further possible requirements.

Concerning to *manipulability*, i.e. the robustness of the method against cheating judges, it is not clear if should be included or not. On one hand, there does not exist any non manipulable method [3] (though there are methods which are more robust than others against cheating [11]). On the other hand, should a discordant verdict be considered as a manipulation?.

Independence of irrelevant alternatives seems a quite reasonable restriction, but it does not hold in the nowadays skating system and neither in many of the ranking systems used ins sports. Since the number of couples in a final is not fixed, it happened more than once that the first classified couples would have changed if the eliminatories would have been more restrictive, and thus leading to complains from the non winner couples. IIR would not be a so important requirement if the number of participants in a final were be fixed or at least larger than it is.

In order to fulfil IIR, it sounds logical that the ranking method should not take into account the absolute positions in the particular rankings as in the *Borda*–like functions (the larger the number of participants, the worse it is to be ranked in the last position), but rather paying attention on a pairwise comparison between couples (counting the number of the judges who prefer couple x to couple y) as in the *Condorcet*–like functions. See [6] for an explanation of Borda and Condorcet methods.

Looking in this direction, Table 2 shows the preferences of judges in pairwise comparison corresponding to Table 1. The entry for cell (i, j) stands for the number of judges who prefer couple i to couple j. We will refer to this table as *pairwise comparison matrix*.

This pairwise comparison often leads to paradoxes. In the example given in Table 2, it is shown that a majority of judges prefer couple 15 to couple 16, and also a majority prefer couple 16 to 17. The paradox is that also a majority prefer couple 17 to couple 15. We will refer to this paradoxes as *Condorcet cycles*.

	11	12	13	14	15	16	17
11	-	2	2	2	2	2	2
12	3	-	3	2	5	4	5
13	3	2	-	2	4	3	5
14	3	3	3	-	4	3	5
15	3	0	1	1	-	3	2
16	3	1	2	2	2	-	3
17	3	0	0	0	3	2	-

Table 2: Pairwise comparison matrix from Table 1

It is not strange the existence of these "inconsistences", because when ranking couples there are different points of view and also different aspects to be considered. This could lead to the existence of Condorcet cycles even in the judges' mind. On the other hand, these cycles are not simply academic pathologies, but are often found in real competitions (specially in the middle ranked couples) (see [11] for examples of cycles in real figure skating olympic competitions).

In 1959, Kemeny proposed a method based on the idea of distance between any two preference orders [4, 5]. The defined distance was obtained as the only definition meeting a set of axioms (including triangular inequality, and other standard properties of distances). The cumulative preference is given by the preference that minimizes either the sum of distances to the rankings given by judges, or either the sum of the squares of such distances. Kemeny let open both possibilities without specifying which one was better.

In 1978, Young and Levenglick [12] proved that the Kemeny method, when minimizing the sum of distances, is the unique preference function that is neutral, consistent and Condorcet (MP). Young also proved that Kemeny method is IIA. Due to that paper, Kemeny method when minimizing the sum of distances is also known as *Kemeny-Young* method.

These previous results apparently solves the original question of X.Mora, setting Kemeny-Young method as the method meeting the proposed requirements (the discrimination requirement is not really a restriction, but a matter of degree). In fact Kemeny-Young has already been proposed as an interesting ranking function for figure skating [11]. (There are other ranking functions satisfying MP and IIA requirements such as the *Copeland* method. The reader is referred to [13] for more information about other functions).

The remaining sections of the paper are devoted to better understanding of ties in the kemeny-Young method as well as some proposals to solve ties.

3 Kemeny-Young method by means of graph theory

Let us call *competition* $C(n, k)$ to a subset with cardinality k of the symmetric group S_n. A permutation in $C(n, k)$ will stand for a sorting of couples done by one judge (a column in Table 1a).

Given a competition $C(n, k) = \{\sigma_1, \ldots, \sigma_k\} \subset S_n$, the *competition graph* is defined as the edge–colored bipartite (multi)–graph $CG(n, k) = (V_1 \cup V_2, E, \phi)$ with $V_i = \{(i, a_j) : j = 1..n\}$, $i = 1, 2$, and $\phi : E \to \{1, 2, \ldots, k\}$, and such that there exists an edge $e = [(1, a_i), (2, a_j)]$ with color $\phi(e) = m$ whenever $\sigma_m(a_j) = a_i$.

Given $\tau \in S_n$, let $c_m(\tau)$ be the number of crossings between edges with the same color m when the graph is drawn as follows: Vertices in V_1 are placed in a straight line sorted from left to right $(1, \tau(a_1)), \dots (1, \tau(a_n))$. Vertices in V_2 are placed in a parallel line sorted $(2, a_1), \dots, (2, a_n)$, and edges are represented by straight lines.

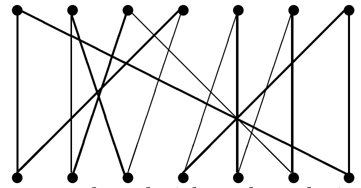


Figure 1: Competition graph for judges B (thinlines) and C (thicklines)

Figure 1 shows a Competition graph for judges B (thinlines) and C (thiklines) of the example in Table 1 drawn considering that τ is the identity permutation. Labels for the vertices in that example illustrate the meaning for the graph: Lower vertices should be seen as ranking positions and upper vertices will stand for couples. In the example of the figure, we have $c_B(Id) = 3$ and $C_C(Id) = 11$.

In general, the value of $c_m(\tau)$ measures the disagreement between the preference given by permutation *tau* and the preference of judge m. It is not difficult to prove the following result:

Proposition 3.1. *Let $d(\tau, \sigma_m)$ be the distance between to permutations τ and σ_m defined by Kemeny. Then $d(\tau, \sigma_m) = 2c_m(\tau)$.*

The *cost* of a given permutation τ with respect to a competition $C(n, k)$ is defined as $d(\tau, C(n, k)) = \sum_{i=1}^{k} d(\tau, \sigma_i)$, and the *solution* of a competition is defined as the set of permutations with minimum cost with respect to the competition, i.e. $\mathcal{S}(C) = \{\tau \in S_n \,|\, d(\tau, C(n, k)) \leq d(\sigma, C(n, k)), \quad \forall \sigma \in S_n\}$. The set $\mathcal{S}(C)$ is clearly the set of collective preferences given by the Kemeny-Young method. (Notice, however, that Kemeny allowed solutions to be ties, i.e. preferences not being permutations, but here we restrict solutions to be strictly permutations).

From the point of view of graph theory, finding the solutions of a given competition is equivalent to solve a slight variation of the *One side crossing minimization problem* (OSCM), i.e. placing the vertices in V_1 on a line in such a way that the number of edge-crossings is minimized. This problem was studied in [10], and we will use the same techniques to better understanding how can the elements in $\mathcal{S}(C)$ be.

4 Sorting graph. Solution for competitions

Following the ideas in [10], let us consider a permutation τ_1 and let τ_2 be another permutation obtained from τ_1 by transposing two adjacent couples a_i and a_j, i.e. $\tau_1 = (\dots, a_i, a_j, \dots)$ and $\tau_2 = (\dots, a_j, a_i, \dots)$. Then, the difference of costs of permutations τ_1 and τ_2 is due to the crossings between edges incident to a_i and the edges incident to a_j.

Precisely, let $G_{i,j}$ be the subgraph of $CG(n, k)$ obtained by deleting all vertices in V_1 except vertices $(1, a_i)$ and $(1, a_j)$. Let $c(i, j)$ and $s(i, j)$ be the number of crossings between edges with the same color, and edges with different colors, respectively, in the graph $G_{i,j}$ when it is drawn as follows: Vertices $(1, a_i)$ and $(1, a_j)$ are placed in a straight line being $(1, a_i)$ placed on the right hand side. Vertices in V_2 are placed in a parallel line sorted $(2, a_1), \ldots, (2, a_n)$, and edges are represented by straight lines. In terms of the original problem, $c(i, j)$ is the number of judges who prefer couple a_i to couple a_j.

Let the *sorting graph* be the directed weighted graph $SG(C) = (V, A, \psi)$, with $V = \{1, \ldots, n\}$, and such that there is an arc (i, j) whenever $c(i, j) > c(j, i)$. The weight of the arc (i, j) is $\psi(i, j) = a_{ij} = c(i, j) - c(j, i)$. Notice that the sorting graph can be obtained immediately from the pairwise comparison matrix.

Since a judge cannot assign the same place in the ranking to two different couples, it is immediate that $c(i, j) + c(j, i) = k$, where k stands for the number of judges, and thus, If k is odd, then $c(i, j) - c(j, i) \neq 0$. This means that given any to vertices i, j in the sorting graph, there will always exist either the arc (i, j) or either the arc (j, i). In other words:

Remark 4.1. If the number of judges is odd, then $SG(C)$ is a tournament.

In case the graph $SG(C)$ is a tournament not containing directed cycles, it is a *total order* that determines univocally a unique solution for $C(n, k)$ ($|\mathcal{S}(C)| = 1$). But if $SG(C)$ is not a tournament (and has neither no directed cycles), it is a partial order. This partial order, in general, is compatible with more than a total order. And hence $|\mathcal{S}(C)| > 1$, i.e. a tie will appear. Since these kind of ties can be avoided by using an odd number of judges, from now on we will assume that the number of judges is odd.

The problem arrises when $SG(C)$ contains Condorcet cycles. However, the existence of Condorcet cycles does not necessary imply a multiple solution in the Kemeny-Young method, and in fact most Condorcet–like methods differ one from each other only in the way they deal with Condorcet cycles.

The removal of cycles by deleting arcs such that the total weight of removed arcs is minimum is known as the *feedback arc–set problem* (FAS) [2]. In [10] it is shown that the minimum number of crossings in $CG(n, k)$ are obtained by solving the FAS problem in the graph $SG(C)$.

Taking into account the following easy result,

Proposition 4.2. *Given a tournament G, each solution of FAS for G produces a partial order, which is compatible with a unique total order.*

it is straightforward that

Theorem 4.3. *If C is a competition with an odd number of judges, then each solution of the FAS problem for $SG(C)$ determines a solution of the Kemeny-Young method for the competition C.*

As a consequence, $\mathcal{S}(C)$ contains more than one permutation if and only if there is more than one solution of the FAS problem for $SG(C)$.

In the example given in Table 1, there is only one Condorcet cycle (couples 15,16 and 17), but the weights of arcs in that cycle are all the same. Hence, there are three different solutions for the FAS problem (each one corresponding to the removal a single arc in the cycle). In that case, a tie must be given to couples 15, 16 and 17. The final ranking should be, then,

$$\begin{array}{ccccc} & & & 15 & \\ 14 & 12 & 13 & 16 & 11 \\ & & & 17 & \end{array}$$

Ties in Kemeny-Young method are not always as "nice" as in this example. In Table 3, there are up to 9 Condorcet cycles (11-13-12; 11-13-15; 11-13-12-14; 11-13-15-14; 11-13-12-15-14; 11-13-12-15; 12-14-13; 12-15-14-13; 13-15-14) and two different solutions for the FAS problem, leading to two possible rankings (13,12,15,14,11 and 12,15,14,11,13). Notice that couple 13 is in a very strange situation.

	A	B	C
11	5	4	1
12	2	1	5
13	1	5	3
14	4	3	2
15	3	2	4

Table 3: Another example

5 Modified Sorting Graph: Tie breaking.

Ties in sports are usually solved by using other secondary criteria to determine the final ranking among possible solutions given by the main sorting method. These secondary criteria take into account different aspects not considered by the main sorting function and this usually leads to a loss of some properties of the global sorting function. Precisely we conjecture the following:

Conjecture 5.1. *There is no possible method to select one permutation among the solutions for a competition preserving the IIA property in all instances.*

Obviously it would make no sense giving more than one ranking as a solution for a sports competition (specially if solutions are like the one shown in the example of Table 3). For this reason we propose a method to solve some of the ties in Kemeny-Young method which appears as a natural refinement from the point of view of graph theory.

In order to remove undecided condorcet cycles, we propose to look for information not taken into account until now: Condorcet–like methods use only preferences of individual judges to build a solution, let us say "vertical information", while information regarding to the rank different judges give to a certain couple has been skipped ("horizontal information").

5.1 First proposal

Let us modify the sorting graph by introducing some "perturbations" on the weights of the arcs. The modified sorting graph $SG' = (V, A, \psi')$ has the same vertex and arc sets that SG, but given an arc (i, j), $\psi'(i, j) = a_{ij} + \epsilon b_{ij}$ where $b_{ij} = s(i, j) - s(j, i)$ and ϵ is such that $\epsilon b_{ij} < 1$, $\forall i, j \in V$.

The perturbations added take into account the discrepancies of judges regarding to the ranking of a given couple with respect to the others. From the graph theory point of view we are not only looking at monochromatic edge–crossings, but to all possible crossings. The condition imposed on ϵ assures that this perturbations will only be considered in those cases in which there is an undecided tie (acting as a secondary function). Some of the cycles can be untied now by using FAS on the modified sorting graph. Following the example in Table 1, the Condorcet cycle has now the weights show in Figure 2

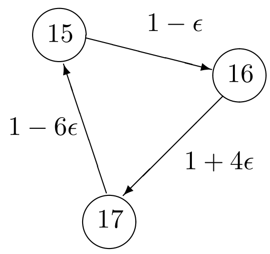


Figure 2: edge weights in the modified competition graph for vertices 15,16 and 17, corresponding to Table 1

and the FAS solution turns to be unique (by removing arc (17,15)), leading to a unique ranking given by

$$11 \quad 12 \quad 13 \quad 15 \quad 16 \quad 17 \quad 11$$

5.2 Second proposal

The idea of this second proposal is to make "corrections" on the rankings given by the judges according to their discrepancies with respect to the average position given to couples. Namely, let us consider ordinal positions as points on a line (like in the competition graph), and let us move those points closer to the mean value for each couple.

Let $\{r_{i1}, \ldots, r_{ik}\}$ be the set of ordinals given to couple c_i (Table 1b) and let $\overline{r_i} = \frac{1}{k}\sum_{j=1}^{k} r_{ij}$ be the mean value of those numbers. Given $0 < \alpha < \frac{1}{2k}$, let $r'_{ij} = r_{ij} - \alpha(r_{ij} - \overline{r_i})$ Then the modified competition graph $CG' = (V'_1 \cup V'_2, E', \phi')$ is defined as the edge–colored bipartite (multi)–graph with $V'_1 = \{c_i : i = 1..n\}$, $V'_2 = \{r'_{ij} : i : 1..n,\, j = 1..k\}$, $E' = \{(c_i, r'_{ij}) : i : 1..n,\, j = 1..k\}$ and $\phi'(c_i, r'_{ij}) = j$. Let us draw vertices in V_2 on a straight sorted from left to right according to their numeric value. Figure 3 shows the modifications introduced in the Competition Graph for a certain $\alpha > 0$. (Notice that if $\alpha = 0$ then $CG = GC'$.)

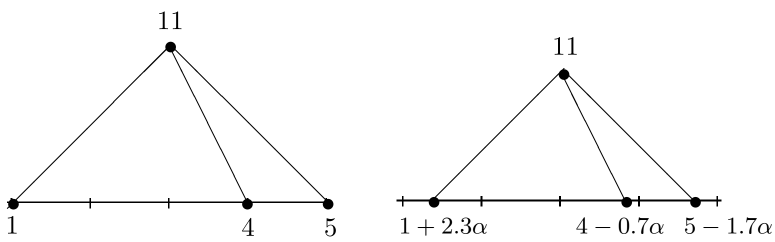


Figure 3: Modifications on GC

The second proposal consists on applying the FAS algorithm on the modified sorting graph associated to the modified competition graph. The upper bound for α ensures that $r'_{ij} > r'_{i'j'}$ if $r_{ij} > r_{i'j'}$ and thus the values for $c(i,j)$ will be the same as in the previous sections.

Notice that if $\alpha = 1$ the method is equivalent to the mean method, i.e. couples are sorted according to the mean value of the ordinals they are given by judges.

The solutions given by this modification are a subset of the solutions for the Kemeny-Young method. This second proposal seems more selective than the proposed in the previous subsection, though both fail to untie the example in Table 3.

6 Conclusions

We presented in this paper a proposal for a ranking method in dancesport competitions. The method is based on the Kemeny-Young method. The interpretation done of this method through graph theory allowed us to find a refinement of the method in the sense that the number of ties is smaller. As a counterpart, under some instances the method is not independent from irrelevant alternatives.

We strongly recommend the number of judges to be odd in all competitions in order to avoid unnecessary ties.

Acknowledgements

The author wants to thank the organizers of the GEMT (Study Groups of Mathematics and Industry), where this problem was proposed. And specially wants to thank the dancesport scrutineer Xavier Mora for his useful comments and suggestions.

References

[1] K.J. Arrow. Social choice and individual values. John Wiley, New York, 1963

[2] F.Gavril. Some NP–complete problems on graphs. In *11th Conf. on Information Sciences and Systems.* John Hopkins University Press. Baltimore. 1977. 91–95.

[3] A. Gibbard. Manipulation of voting schemes: a general result. *Econometrica*, 41 (1973), 587–601.

[4] J. Kemeny. Mathematics without numbers. *Dedalus*, 88 (1959), 571–591.

[5] J. G. Kemeny and J. L. Snell. Preference rankings. An axiomatic approach. In *Mathematical Models in the Social Sciences.* Blaisdell Publishing, 1962. (Chapter 11).

[6] I. McLean, A.B.Urken (eds.), *Classics of Social Choice.* University of Michigan Press, Ann Arbor, 1995

[7] X. Mora. Improving the skating system (II). Methods and paradoxes from wider perspective. `http://puffinet.com/escrutini/iss2en.pdf` (2002)

[8] X. Mora. The Skating System. `http://puffinet.com/escrutini/skating1en.pdf`

[9] X. Mora. Puntuació en competicions amb diversos jutges. in *Groups d'estudi de matemàtiques i tecnologia (GEMT).* UPC, 2002

[10] X. Muñoz, W. Unger and I. Vrto. One side crossing minimization is NP-hard for sparse graphs. In *Graph Drawing. Lecture Notes in Computer Science 2265.* Springer Verlag. 2001. pp. 115–123.

[11] M. Truchon, Figure skating and the theory of social choice. *Cahier n.9814*. CREFA. Université Laval. Quebec. 1998.

[12] H.P.Young and A.Levenglick. A consistent extension of Condorcet's election principle. *SIAM J. of Appl. Math.*, 35 (1978), 285–300.

[13] `http://www.condorcet.org`

Fundamental Antichains

M. Murphy
St Andrews, Scotland

1 Summary

Involvement is a partial well order on the set of all finite permutations. Pattern Classes are the down closed sets of permutations under this partial order. It is of interest whether a given pattern class is partially well ordered, or in other words if a given pattern class does not contain any infinite antichain. We approach this question by looking at definitions of infinite antichains that are in some way minimal, and do not involve more patterns than necessary. Especially, we present the very simple definition of a 'fundamental' antichain. Although little is as yet known about fundamental antichains, it is already clear that fundamental antichains produce some very nice patterns, and that there are strong similarities between fundamental antichains in pattern classes, and in topological minors, finite posets under inclusion, and other naturally occuring partial orders.

2 Involvement as a partial order

We give a brief recap of the partial order central to permutation patterns.

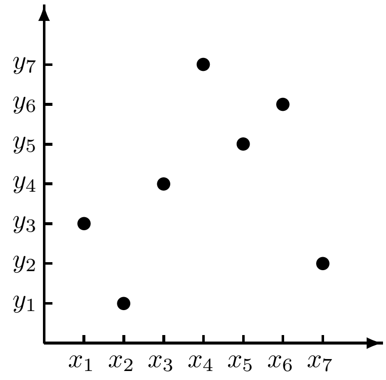
Let A be a totally ordered set, which is to say that if $a, b \in A$ then either $a < b$ or else $a > b$ or else $a = b$. For our purposes A may be thought of without error as any subset of the real numbers. Two sequences $a_1 \ \dots \ a_m$ and $b_1 \ \dots \ b_n$ are said to be *order isomorphic* if they are of equal length, and that being satisfied, if for all $i, j \leq n$ we have that $a_i \leq a_j$ if and only if $b_i \leq b_j$. A typical example of two order isomorphic sequences is given by 1 3 2 4 and 7 27 10 28.

One sequence, $\alpha = a_1 \ \dots \ a_m$, is said to be *involved* in another, $\beta = b_1 \ \dots \ b_n$, if the former is order isomorphic to a subsequence of the latter. If this is the case we may write $\alpha \preceq \beta$. If we call a sequence a permutation if it contains each of the integers $1, 2, \dots, m$ exactly once, then involvement forms a partial well order on the set of all permutations.

Involvement occurs naturally in sorting devices limited by fixed properties, such as limited memory or a restrictive set of rules according to which sorting may occur, as is the case in the card game of Patience. In general, if such a system can sort 317624958 then it can sort every sequence involved in that permutation, such as 3142. The study of the involvement partial order is applicable to situations in parallel computing and data communication networks.

3 A notation with potential

In common with many students of restricted permutations we like to plot permutations on two dimensional coordinates. The plotting method that we prefer to use, by M.D. Atkinson and R. Beals has a nice consequence for the pattern classes that cannot be written as a union of two smaller pattern classes, and so we afford it a brief comment.



The figure shows a typical plot of the permutation $\pi = 3\ 1\ 4\ 7\ 5\ 6\ 2 = \pi(1)\ \ldots\ \pi(7)$. It is clear that the x-coordinates satisfy $x_1 < x_2 < \ldots < x_7$, the y-coordinates $y_1 < y_2 < \ldots < y_7$ and that if the point with x-coordinate x_i is denoted $f(x_i)$ (where $i \in \{1, 2, \ldots, 7\}$) then $f(x_i) = y_{\pi(i)}$. We require nothing more than this of the plot. We do not require the coordinates to be integers, nor do we require the values on the x-coordinate to be the same, permuted, as those on the y. And yet the plot uniquely identifies a permutation.

The leniency of this notation causes that if we have any injective function $f : A \to B$ from one subset A of the real numbers to another, B, then any restriction of f to a finite domain represents a permutation. Given such a function, the set of all permutations represented by restricting it to a finite domain is clearly down closed, and so is a pattern class. Finally, as the coup de bonheur, we have that the set of all pattern classes representable in this way is precisely the set of pattern classes that cannot be expressed as the union of two smaller pattern classes. Thus pattern classes indivisible by finite union can be thought of in terms of a function plotted on two axes.

Some nice results about union indivisibility and down closed subsets of arbitrary partial orders can be found in R. Diestel and O. Pikhurko's paper [3] (On the cofinality of infinite partially ordered sets: factoring a poset into lean essential subsets), which refers to earlier work. But we finish our notational comment by giving a simple example of a pattern class not expressible as the union of two smaller pattern classes, and a real function that represents it. The down closed pattern class $X = Sub(1 \oplus R) = \{12, 132, 1432, 15432, \ldots\} \cup \{1, 21, 321, 4321, \ldots\}$, and the function $f : [0, 1) \to [0, 1)$ defined by:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 - x & \text{otherwise} \end{cases}$$

The function f restricted to finite domains including zero represents the permutations $\{12, 132, 1432, 15432, \ldots\}$, and f restricted to finite domains excluding zero represents the permutations $\{1, 21, 321, 4321, \ldots\}$.

Note: The pattern classes arising from functions on some domains and ranges have been classified; results are available in [2].

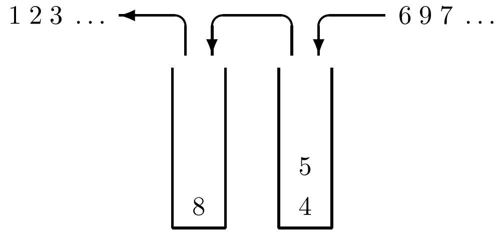
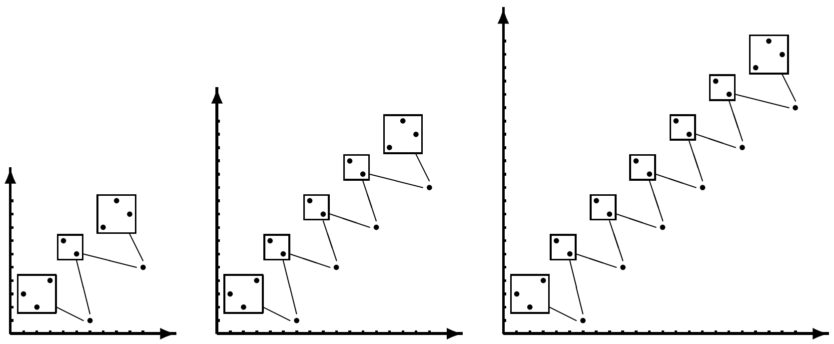


Figure 1: Two stacks in series, a simple sorting network.

4 The main subject: Simple infinite antichains

We will demonstrate that infinite antichains do appear in natural permutation class contexts, that they are important, and that they can still be both simple and pleasing.

We define the antichain $X1$, the first three terms of which are plotted:



In general elements of $X1$ have the form:

$$\xi_n = 3\,2\,4\,7\,6\,1\;\;10\,9\,5\;\;13\,12\,8\;\;16\,15\,11\;\;19\,18\,15\;\;\ldots$$
$$\ldots 6n+1\;6n\;6n-4\;\;6n+2\;6n+4\;6n+3\;6n-1$$

where $n \in \mathbb{Z}^+$.

It is not elementary to see, but can still be proved by a short and elegant argument, that none of the permutations in $X1$ can be sorted by two stacks in series. However they are minimal in that respect as every permutation involved in, but not equal to, an element of $X1$ can be sorted. That also is easy to prove, but to give a specific example, let the reader try as he will to sort the permutation $\xi_1 = 3\ 2\ 4\ \ 7\ 6\ \ 1\ \ 8\ 10\ 9\ \ 5$. Now let the reader attempt to sort the permutation $\xi_1 \setminus \{3\} \cong 2\ 3\ \ 6\ 5\ \ 1\ \ 7\ 9\ 8\ \ 4$, which is order isomorphic to the sequence ξ_1 less the first term. The previously frustrated effort will succeed quite easily, as it would were any other term of ξ_1 removed.

Of the two most basic antichains of pattern classes, namely minimal permutations not in a class (basis) and maximal permutations within a class, this demonstrates an example of the first kind even though we do not give the entire basis. It is sufficient to note that infinite antichains do appear in natural situations and that a better understanding of infinite antichains will aid the understanding of pattern classes as a whole. We will therefore attempt to isolate the essence of permutation antichains.

5 Superfluous terms

We will consider an infinite antichain, $X2$, and contrast it with another by M. Bona. It will be clear that the permutations in $X2$ contain terms that do not contribute to $X2$ being an antichain. We define an antichain to be *trim* if no permutation in it contains such superfluous terms:

Definition 5.1. *An antichain Z is* trim *if there does not exist $\zeta \in Z$ and a permutation $\eta \prec \zeta$ such that $(Z \setminus \{\zeta\}) \cup \{\eta\}$ is an antichain.*

Note: Every non-trim antichain can be reduced to a trim antichain of the same size, by removal of terms. This process does not necessarily produce a unique result.

6 Terms supporting superfluous sub-antichains

We will consider another infinite antichain, $X3$, that is trim but that still contains permutations with superfluous terms of another sort: Terms necessary to maintain some relatively small subset of $X3$ in the state of being an antichain, but where the complement of that small part still contains an infinite antichain.

Definition 6.1. *An antichain Z is* strongly trim *if there does not exist a permutation η properly involved in an element of Z, and a subset Y of Z such that $Y \cup \{\eta\}$ is an antichain having the same size as Z.*

Note: Every non strongly trim antichain can be reduced to a strongly trim antichain of the same size, by removal of terms and entire permutations. This process does not necessarily produce a unique result.

7 Fundamental antichains

There are infinite permutation antichains, strongly trim, constructed by *wreath* from another infinite antichain of the same cardinality. Wreath is a permutation construction introduced in [1], that gives rise to the divisibility ordering and RIP-tree frames of [2]. Since constructing infinite antichains from other infinite antichains and a divisibility construction is an understood science, this suggests that if we wish to seek the basic building blocks of infinite antichains, we need a definition yet more selective than that of strongly trim. We suggest the following:

Definition 7.1. *An antichain Z is* fundamental *if the down closure of Z under involvement contains no antichains Y of the same size as Z, except those that are subsets of Z itself.*

We propose that fundamental antichains are those most worthy of examination. This has numerous reasons that we sketch out below.

All fundamental antichains known at present have very simple, pleasing and regular patterns. Fundamental antichains are defined in a poset theoretic context, and infinite fundamental antichains also exist for finite graphs under deletion of vertices, finite posets under the partial order of inclusion, and topological minors. In all of these contexts simple regular patterns appear with strong and obvious similarities with the permutation antichains. In the case of finite graphs under deletion of vertices, all fundamental antichains have been classified. It seems entirely feasible to do the same for fundamental permutation antichains, even though it is known that there are infinitely many fundamental permutation antichains.

Fundamental antichains behave nicely in terms of established well known pattern class concepts. To give an example we first define the *proper down closure* of a permutation ζ to be all permutations involved in ζ, but excluding ζ itself. Proper down closure extends easily to sets of permutations, providing that the permutations form an antichain. The proper down closure of an infinite antichain is a pattern class, and one that cannot be expressed as the union of two smaller pattern classes. Recall the reason for our plotting notation. Another example of nice behaviour is this: An antichain Z is said to be *maximal* if there does not exist a permutation $\zeta \notin Z$ such that $Z \cup \{\zeta\}$ is an antichain. It is conjectured that the down closure of every infinite maximal fundamental antichain is finitely based. This conjecture has been tested on all the simpler known infinite fundamental antichains, and is believed to be true for all known infinite fundamental antichains.

Finally, in almost every sphere of structural analysis of permutation patterns it is of interest whether a given class is partially well ordered, i.e. does not contain an infinite antichain. It is easy to see that a pattern class contains an infinite antichain if and only if it contains an infinite fundamental antichain. So a classification of all infinite fundamental antichains would solve the partial well ordered question in all cases.

8 Current analysis of fundamental antichains

The approach we are using at present is to regard wreath decomposition as a divisibility order on permutations, and to use wreath in this context to examine fundamental antichains. The RIP-frames of [2] are of use.

Theorem 8.1. *Only finitely many permutations in a fundamental antichain have a wreath interval of size greater than two.*

Conjecture 8.2. *Every permutation in an infinite fundamental antichain has either no, one or two maximal intervals of size two.*

We expect that fundamental antichain results for permutations will have analogous results for fundamental antichains in other contexts, such as graph minors.

References

[1] M.D. Atkinson, T. Stitt: "Restricted permutations and the wreath product", Discrete Math. 259 (2002) 19-36.

[2] M.M. Murphy (with grateful thanks to help from N. Ruškuc and M.D. Atkinson): "PhD Thesis: Restricted Permutations, Antichains, Atomic Classes and Stack Sorting", submitted to St Andrews University, Scotland, 2002.

[3] R. Diestel, O. Pikhurko: "On the cofinality of infinite partially ordered sets: factoring a poset into lean essential subsets", (preprint April 2002).

The algorithmic hardness of the sub-permutation problem

Yuri Rabinovich
Dept. of Computer Science
Haifa University

We study various computational aspects of the problem of determining whether a given order contains a given sub-order. Formally, given a permutation π on k elements, and a permutation σ on $n > k$ elements, the goal is to determine whether there exists a strictly increasing function f from $[1..k]$ to $[1..n]$ which is order preserving, i.e. f satisfies $\sigma(f(i)) > \sigma(f(j))$ whenever $\pi(i) > \pi(j)$. We call this decision problem the sub-permutation problem.

The study falls into two parts. In the first part we develop and analyze an algorithm (or rather an algorithmic paradigm) for this problem. We show that the complexity of this algorithm is at most $O(n^{1+C(\pi)})$ where $C(\pi)$ is a naturally defined function of the permutation π.

In the second part we study $C(\pi)$. In particular, we show that $C(\pi) \leq 0.46k + o(k)$ implying that the complexity of the sub-permutation problem is $O(c_k + n^{0.46k+o(k)})$. On the other hand we prove that for most π's, $C(\pi) = \Omega(k)$ establishing a lower bound for the algorithm. In addition we develop a fast polylogarithmic approximation algorithm for computing $C(\pi)$ and bound the value of this parameter for some interesting families of permutations.

The results are joint work with Shlomo Ahal.

The SW conjecture: a dynamic systems approach

Yuri Rabinovich
Dept. of Computer Science
Haifa University

We develop a new (dynamical system) approach to the Stanley-Wilf (SW) conjecture, and make the first steps in its study. We believe that the new approach will eventually lead to new significant results and to a refined understanding of the circle of problems related to the SW conjecture.

Let π be a fixed forbidden permutation. Consider an infinite tree T whose root is an empty permutation and whose nodes are the π-avoiding permutations σ. The children of $\sigma \in S_{k-1}$ are all the π-avoiding extensions of σ by the new element k. We introduce an equivalence relation $\equiv$ on π-avoiding permutation by $\sigma \equiv \tau$ if the subtrees of T rooted at σ and τ are isomorphic. This leads to a definition of the *type* $[\sigma]$. The object of our study is the linear dynamical system D defined by $[\sigma] \to \{[\sigma']\}$ where $\{\sigma'\}$ are the children of σ in T.

The SW conjecture turns out to be equivalent to the existence of a nonnegative eigenvector of D with positive finite eigenvalue r. Or equivalently, to the existence of a nonnegative *subinvariant potential* p on the types so that:

$$R \cdot p([\sigma]) \geq \sum p(\sigma')$$

for some $R > 0$. We demonstrate the efficiency of this approach by (re-)proving the SW conjecture for special π.

Considering an altered dynamical system (a Markov chain) P where $[\sigma]$ passes to its random child $[\sigma']$, we conclude that SW is equivalent to showing that:

$$\mathbf{E}[X_0 \cdot X_1 \cdots X_{k-1}]^{1/k} \to r < \infty$$

where X_i is the number of children of the ith descendant of (any) σ_0. We make a first step in the desired direction by showing that:

$$\mathbf{E}[X_n] \to c\ (\leq |\pi|^2).$$

Permutation diagrams and forbidden patterns

Astrid Reifegerste*

1. Introduction

In recent time much work has been done counting permutations with restrictions on the patterns they contain. Given a permutation $\pi \in \mathcal{S}_n$ and a permutation $\tau \in \mathcal{S}_m$, an *occurrence of* τ *in* π is an integer sequence $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ such that the letters of the word $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ are in the same relative order as the letters of τ. In this context, τ is called a *pattern*. If there is no occurrence we say that π *avoids* τ, or alternatively, π is τ*-avoiding*.
In [2, 3] we have utilized the diagram of a permutation to study certain forbidden patterns. For $\pi \in \mathcal{S}_n$, we obtain its *diagram* from the $n \times n$ array representation of π by shading, for each dot, the cell containing it and the squares that are due south and due east of it. Each square left unshaded we call a *diagram square*. The connected components of the diagram squares form Young diagrams. For any diagram square, its *rank* is defined as the number of dots northwest of it. (Clearly, connected diagram squares have the same rank.)
In this paper, we consider the both permutation statistics which count the distinct pairs arising from the last two terms of occurrences of patterns $\tau_1\cdots\tau_{m-2}m(m-1)$ and $\tau_1\cdots\tau_{m-2}(m-1)m$ in a permutation, respectively. By a simple involution in terms of permutation diagrams we will prove their equidistribution over the symmetric group. As special case we obtain a one-to-one correspondence between permutations which avoid each of the patterns $\tau_1\cdots\tau_{m-2}m(m-1) \in \mathcal{S}_m$ and such ones which avoid each of the patterns $\tau_1\cdots\tau_{m-2}(m-1)m \in \mathcal{S}_m$ which coincides for $m = 3$ with the bijection given by Simion and Schmidt in their famous paper about restricted permutations.

2. Main Results

For $m \geq 2$, define the pattern sets

$$\begin{aligned} A_m &= \{\tau \in \mathcal{S}_m : \tau_{m-1} = m,\, \tau_m = m-1\} \\ B_m &= \{\tau \in \mathcal{S}_m : \tau_{m-1} = m-1,\, \tau_m = m\}. \end{aligned}$$

For a permutation $\pi \in \mathcal{S}_n$, denote by $\mathsf{a}_m(\pi)$ resp. $\mathsf{b}_m(\pi)$ the number of distinct pairs (i, j) where $1 \leq i < j \leq n$ such that i, j are the last terms of an occurrence of a pattern belonging to A_m and B_m, respectively, in π. If we have $\mathsf{a}_m(\pi) = 0$ then π avoids each pattern in A_m; analogously, $\mathsf{b}_m(\pi) = 0$ means the avoidance of every pattern of B_m.
For example, in $\pi = 7\,1\,4\,2\,6\,3\,5 \in \mathcal{S}_7$ the sequences $(2,3,5,7), (2,4,5,6), (2,4,5,7)$ are the occurrences of 1243, $(3,4,5,7)$ is the only occurrence of 2143. Furthermore, π contains once the pattern 1234 and avoids 2134. Hence $\mathsf{a}_4(\pi) = 2$ and $\mathsf{b}_4(\pi) = 1$.

It is easy to determine the number $a_m(\pi)$ from the ranked diagram of π. Obviously, any square (i, j) is a diagram square of rank at least $m-2$ if and only if i, π_j^{-1} are the last two terms of an occurrence of a pattern contained in A_m.

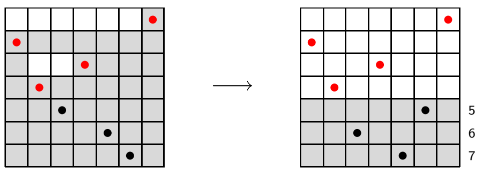
Proposition 1. *Let $\pi \in \mathcal{S}_n$ be a permutation. Then $\mathsf{a}_m(\pi)$ equals the number of diagram squares of rank at least $m-2$. In particular, π avoids each pattern in A_m if and only if every diagram square is of rank at most $m-3$.*

*Institut für Mathematik, Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany
email: reifegerste@math.uni-hannover.de

By definition, the number $\mathsf{b}_m(\pi)$ counts the number of non-inversions on the letters of π which are greater than at least $m-2$ elements to its left. (Here a pair (i,j) is called a *non-inversion* if $i<j$ and $\pi_i<\pi_j$.) It is easy to see as well that all informations about a permutation are encoded in the diagram squares of rank at most $m-3$ and the set of pairs which are counted by $\mathsf{b}_m(\pi)$.

Proposition 2. *Any permutation $\pi \in \mathcal{S}_n$ can be completely recovered from the diagram squares having rank at most $m-3$ and the pairs (i,j) arising from the last two terms in an occurrence of a pattern belonging to B_m.*

Let D be a set of squares, and let O be a set of integer pairs that are obtained from a permutation $\pi \in \mathcal{S}_n$ as diagram squares of rank at most $m-3$ and as ending terms of occurrences of patterns belonging to B_m, respectively. Then we can recover π as follows:
First represent the elements of D as white squares in an $n \times n$ array, shaded otherwise. Row by row, put a dot in the leftmost shaded square such that there is exactly one dot in each column. Each dot having at least $m-2$ dots northwest is deleted now. Then arrange the missing dots such that O is precisely the set of non-inversions on these dots, that is, the dot contained in the ith row lies strictly to the left of the dot contained in the jth row if and only if $(i,j) \in O$.
For example, the diagram squares of rank at most 1 of the permutation $\pi = 7\,1\,4\,2\,6\,3\,5 \in \mathcal{S}_7$ are $(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(3,2),(3,3)$. Furthermore, the only occurrence of a pattern of B_4 in π, namely $(2,4,6,7)$, ends with $(6,7)$. Thus we obtain:

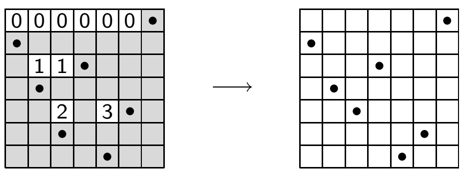


These properties of permutation diagrams are fundamental for the construction of a bijection Φ_m on the symmetric group which proves

Theorem 3. *We have $|\{\pi \in \mathcal{S}_n : \mathsf{a}_m(\pi)=k\}| = |\{\pi \in \mathcal{S}_n : \mathsf{b}_m(\pi)=k\}|$ for all n and k.*

Let $\pi \in \mathcal{S}_n$ and $D(\pi)$ the set of its ranked diagram squares. Define $\Phi_m(\pi)$ to be the permutation whose diagram squares of rank at most $m-3$ are just the elements of $D(\pi)$ of rank at most $m-3$ and whose pairs (i,j) arising from the last two terms of an occurrence of any pattern of B_m are just those pairs for which (i,π_j) is an element of $D(\pi)$ of rank at least $m-2$. By the construction, we have both $\mathsf{b}_m(\Phi_m(\pi)) = \mathsf{a}_m(\pi)$ and $\mathsf{a}_m(\Phi_m(\pi)) = \mathsf{b}_m(\pi)$ for all $\pi \in \mathcal{S}_n$.

For example, for $\pi = 7\,1\,4\,2\,6\,3\,5 \in \mathcal{S}_7$ we obtain $\Phi_4(\pi) = 7\,1\,4\,2\,3\,6\,5$:



The diagram squares having rank at most 1 coincide for π and $\Phi_4(\pi)$. The set O containing the ending terms of the occurrences of patterns of B_4 is obtained from the diagram squares of rank at least 2. (We have $O = \{(5,6),(5,7)\}$.) We have $\mathsf{a}_4(\pi) = \mathsf{b}_4(\Phi_4(\pi)) = 2$ and $\mathsf{b}_4(\pi) = \mathsf{a}_4(\Phi_4(\pi)) = 1$.

The case $k = 0$ means the *Wilf-equivalence* of the pattern sets A_m and B_m, that is, there are as many permutations in $\mathcal{S}_n$ which avoid each pattern in A_m as those which avoid each pattern in B_m. Their number was determined in [1].

Corollary 0.3. *For each $m \geq 2$, the sets A_m and B_m are Wilf-equivalent.*

3. Some remarks

Restricting Φ_m on the set of permutations which avoid each pattern in A_m yields a bijection between A_m-avoiding and B_m-avoiding permutations. For $\pi \in \mathcal{S}_n$ with $\mathsf{a}_m(\pi) = 0$, the construction of $\sigma = \Phi_m(\pi)$ is even more simple. Every diagram square of π has rank at most $m-3$. Therefore the construction works as follows. Set $\sigma_i = \pi_i$ if there are at most $m-3$ integers $j < i$ satisfying $\pi_j < \pi_i$. Then arrange the remaining elements in decreasing order.
In case $m = 3$ this describes exactly the bijection between 132-avoiding and 123-avoiding permutations given by Simion and Schmidt in [4, Prop. 19].

Moreover, the restriction of Φ_m on the set of A_m-avoiding permutations is reasonable regarding some further permutation statistics.

Proposition 4. *Let $\pi \in \mathcal{S}_n$ be a permutation which avoids each pattern of A_m, and $\sigma = \Phi_m(\pi)$. Then the number of elements π_i having at least $m-2$ elements π_j to its left equals the number of elements σ_i satisfying $i + \sigma_i > n + m - 2$.*

References

[1] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, Permutations avoiding an increasing number of length-increasing forbidden subsequences, *Discrete Math. Theor. Comput. Science* **4** (2000), 31-44.

[2] A. Reifegerste, On the diagram of 132-avoiding permutations, preprint, arXiv:math.CO/0208006.

[3] A. Reifegerste, On the diagram of Schröder permutations, preprint, arXiv:math.CO/0209227.

[4] R. Simion, F. W. Schmidt, Restricted Permutations, *Europ. J. Combinatorics* **6** (1985), 383-406.

Patterns in Partitions

Bruce E. Sagan
Michigan State University
E. Lansing, MI 48824
sagan@math.msu.edu
www.math.msu.edu/˜sagan

1 Introduction

The field of patterns in permutations is a very rapidly growing area with connections to combinatorics, computer science, and algebraic geometry. Our purpose is to define an analogous notion of patterns in set partitions which promises to be as rich a vein for exploration. In fact, we will give two possible set-partition analogues of this concept both of which have interesting consequences. But first we begin with some definitions and notation.

Given any sets S, T and a function $f : S \to T$, we apply f element-wise to combinatorial objects built from S. So if $p = a_1 a_2 \dots a_k$ is a word over the alphabet S (possibly with repeated elements) then we let

$$f(p) = f(a_1)f(a_2)\dots f(a_k). \tag{1.1}$$

Another example that will be useful is if we take a subset $B \subseteq S$ then

$$f(B) = \{f(a) \mid a \in B\}.$$

One function that we will be particularly interested in is the restriction map. Given positive integers $m, n \in \mathbb{P}$ we let $[m, n] = \{m, m+1, \dots, n\}$ and if $m = 1$ then we write $[n] = [1, n]$. Now given any set $S \subset \mathbb{P}$ of cardinality $\#S = n$, the unique order-preserving function $r_S : S \to [n]$ will be called the *restriction map of S*. For example, if $S = \{2, 5, 7, 8\}$ then the corresponding restriction map is $r_S(2) = 1,\ r_S(5) = 2,\ r_S(7) = 3,\ r_S(8) = 4$.

In this context, we can formulate the idea of pattern containment in permutations as follows. Let q be a pattern, i.e., q is a permutation in some $\mathfrak{S}_k$, the symmetric group on $[k]$. Then $p = a_1 a_2 \dots a_n \in \mathfrak{S}_n$ *contains* q if there is a subword $p' = a_{i_1} a_{i_2} \dots a_{i_k}$ of p such that

$$r_S(p') = q$$

where $S = \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$. Both of our definitions of pattern containment in set partitions will be of this form.

2 The first definition

Let us start with some basic defintions concerning set partitions. A *partition* π *of a set* S is a family of nonempty sets $B_1, B_2, \ldots, B_l$ such that $\uplus_i B_i = S$ (disjoint union). The B_i are called the *blocks* or *parts* of π and the number of blocks or *length of* π is denoted $l = l(\pi)$. We will write $\pi = B_1/B_2/\ldots/B_l \vdash S$ and will leave out the set braces and commas from the B_i to reduce clutter. So, for example, $137/28/456/9 \vdash [9]$.

The set of all partitions of $[n]$ will be denoted Π_n and let $\Pi = \uplus_{n\geq 1}\Pi_n$. Now given two partitions $\pi = B_1/\ldots/B_l$ and $\sigma = C_1/\ldots/C_k$ then we say that σ *is contained in* π, $\sigma \subseteq \pi$, if the blocks of π can be labeled in such a way that $C_i \subseteq B_i$ for $1 \leq i \leq k$. As an example, $28/3/46 \subseteq 137/28/456/9$ because $28 \subseteq 28$, $3 \subseteq 137$, and $46 \subseteq 456$.

Definition 2.1. Let σ be a partition in Π_k, called the pattern. Then $\pi \in \Pi_n$ *contains the pattern* σ, written $\sigma \sqsubseteq \pi$, if there is a partition $\pi' = B'_1/\ldots/B'_k \subseteq \pi$ with

$$r_S(\pi') = \sigma$$

where $\pi' \vdash S$ and $r_S(\pi') = r_S(B'_1)/\ldots/r_S(B'_k)$.

By way of illustration, consider the pattern $\sigma = 13/2$. Then those π containing σ are exactly the partitions which contain three elements with the smallest and largest in one block and the middle one in a different block. In particular, the permutation $\pi = 14/236/5$ contains six copies of $13/2$ corresponding to the subpartitions $14/2$, $14/3$, $26/4$, $26/5$, $36/4$, $36/5$.

We will say that π *avoids* σ if $\sigma \not\sqsubseteq \pi$. Let

$$\Pi_n(\sigma) = \{\pi \in \Pi_n \mid \pi \text{ avoids } \sigma\}$$

and

$$\Pi(\sigma) = \biguplus_{n\geq 1} \Pi_n(\sigma).$$

3 Enumerative results

As might be expected, many of the enumerative results about pattern avoidance for partitions are best expressed in terms of exponential generating functions. So it will be useful to have the following notation

$$\begin{aligned}
\exp(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!}\\
\overline{\exp}(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n!}\\
\exp_k(x) &= \sum_{n=0}^{k} \frac{x^n}{n!}\\
\overline{\exp}_k(x) &= \sum_{n=1}^{k} \frac{x^n}{n!}
\end{aligned}$$

Also, given any formal power series $F(x)$ with real coefficients, we let

$$\mathcal{C}_n F(x) = \text{ the coefficient of } x^n/n! \text{ in } F(x).$$

More generally, given any set S of nonnegative integers we let

$$\mathcal{C}_S F(x) = \sum_{n \in S} \mathcal{C}_n F(x).$$

When dealing with permutations, one has the action of the full dihedral group of the square to cut down on the number of cases one needs to consider. Unfortunately, when dealing with partitions only one symmetry survives. Given $\sigma = C_1/\ldots/C_l \vdash [k]$ we let

$$\sigma' = C_1'/\ldots/C_l' \text{ where } C_i' = \{k+1-a \mid a \in C_i\} \text{ for } 1 \le i \le l. \tag{3.1}$$

For example, if $\sigma = 135/24/6$ then $\sigma' = 642/53/1$. The following lemma follows easily from the definitions.

Lemma 5. *We have*

$$\begin{aligned}
\Pi(\sigma') &= \{\pi' \mid \pi \in \Pi(\sigma)\}, \text{ and}\\
\#\Pi_n(\sigma') &= \#\Pi_n(\sigma) \text{ for } n \ge 1.
\end{aligned}$$

We start by considering the two simplest patterns which are the unique minimal element $\hat{0}_k = 1/2/\ldots/k$ and unique maximal element $\hat{1}_k = 12\ldots k$ of Π_k when partially ordered by refinement. The next result is a direct consequence of the famous Exponential Formula for exponential generating functions [4] and shows a pleasing duality bewteen these two cases.

Theorem 3.1. *For the pattern $\sigma = 1/2/\ldots/k$ we have*

$$\begin{aligned}
\Pi(1/2/\ldots/k) &= \{\pi \mid \pi \text{ has fewer than } k \text{ blocks}\},\\
\#\Pi_n(1/2/\ldots/k) &= \mathcal{C}_n \exp_{k-1}(\overline{\exp}\, x).
\end{aligned}$$

For the pattern $\sigma = 12\ldots k$ we have

$$\begin{aligned}
\Pi(12\ldots k) &= \{\pi \mid \text{each block of } \pi \text{ has fewer than } k \text{ elements}\},\\
\#\Pi_n(12\ldots k) &= \mathcal{C}_n \exp(\overline{\exp}_{k-1} x).
\end{aligned}$$

Although for general k there is no simple expression for the desired coefficients in the previous theorem, things do simplify if $k = 3$. If all the blocks of σ have one or two elements then we will call σ an *involution*. If σ has all blocks of size two then it will be called a *matching*. If $\sigma \vdash S$ is a matching then $|S| = 2i$ for some integer i and the number of such matchings is the double factorial

$$(2i)!! = 1 \cdot 3 \cdot 5 \cdots (2i - 1).$$

Finally, we let the binomial coefficient $\binom{n}{i}$ be zero if $i > n$ or $i < 0$.

Corollary 3.1. *For the pattern* $\sigma = 1/2/3$ *we have*

$$\begin{aligned} \Pi(1/2/3) &= \{\pi \mid \pi \text{ has at most 2 blocks}\}, \\ \#\Pi_n(1/2/3) &= 2^{n-1}. \end{aligned}$$

For the pattern $\sigma = 123$ *we have*

$$\begin{aligned} \Pi(123) &= \{\pi \mid \pi \text{ is an involution}\}, \\ \#\Pi_n(123) &= \sum_{i \geq 0} \binom{n}{2i} (2i)!! \end{aligned}$$

We next consider the two-block pattern $\sigma = 12\ldots(k-1)/k$. In order to explain the partitions avoiding this pattern, we need the following notion. Given a partition $\pi = B_1/\ldots/B_l \in \Pi_n$ and $j \geq 1$, the *intersection of* π *with* $[j]$ is the partition

$$\pi \cap [j] = D_1/\ldots/D_l$$

where $D_i = B_i \cap [j]$ and empty intersections have been deleted.

Theorem 3.2. *we have* $\pi \in \Pi_n(12\ldots(k-1)/k)$ *if and only if there is some* $j \geq 1$ *such that*

1. $\pi \cap [j]$ *has the same length as* π *with all blocks containing at most* $k - 2$ *elements, and*
2. $[j + 1, n]$ *is a subset of a single block of* π.

Furthermore, letting a vertical bar after a function denote evaluation,

$$\#\Pi_n(12\ldots(k-1)/k) = \mathcal{C}_{[n-1]} \frac{\partial}{\partial t} \exp(t\ \overline{\exp}_{k-2}x)|_{t=1} + \mathcal{C}_n \exp(\overline{\exp}_{k-2}x).$$

Again, there are simplifications when $k = 3$.

Corollary 3.2. *For the pattern* $\sigma = 12/3$ *we have*

$$\begin{aligned} \Pi(12/3) &= \{\pi \mid \text{for some } j,\ \pi \cap [j] = \hat{0}_j \text{ and } [j+1, n] \text{ are all in one block of } \pi\}, \\ \#\Pi_n(12/3) &= \binom{n}{2} + 1. \end{aligned}$$

In view of the previous results and the fact that $1/23 = (12/3)'$, we will have completed the enumeration of partitions avoiding a single pattern of length at most three as soon as we do the case $\sigma = 13/2$. Call a partition π *layered* if it is of the form

$$\pi = [1, i]/[i + 1, j]/ \dots /[k + 1, n]$$

The analogous concept for permutations as discussed in [1, 2, 3, 5, 6, 7] has proven very useful.

Theorem 3.3. *For the pattern $\sigma = 13/2$ we have*

$$\begin{aligned} \Pi(13/2) &= \{\pi \mid \pi \textit{ is layered}\}, \\ \#\Pi_n(13/2) &= 2^{n-1}. \end{aligned}$$

4 The Stanley-Wilf Conjecture

Given a pattern permutation $q \in \mathfrak{S}_k$, let $\mathfrak{S}_n(q)$ denote the set of all permutations in $\mathfrak{S}_n$ avoiding q. It will also be convenient to let absolute value signs be an alternate notation for cardinality. The famous Stanley-Wilf conjecture has been one of the stimuli for the interest in permutation pattern avoidance .

Conjecture 4.1 (Stanley-Wilf). For every pattern q the limit

$$\lim_{n\to\infty} |\mathfrak{S}_n(q)|^{1/n}$$

exists and is finite.

Here we will investigate the analogous question for partitions. But it is easy to find patterns σ such that the limit

$$\lim_{n\to\infty} |\Pi_n(\sigma)|^{1/n} \tag{4.1}$$

is infinite. For example, consider $\sigma = 123$ with n even. Then $|\Pi_n(123)|$ is bounded below by the number of matchings of $[n]$. Using Stirling's approximation gives

$$|\Pi_n(123)|^{1/n} \geq n!!^{1/n} = \left[\frac{n!}{2^{n/2}(n/2)!}\right]^{1/n} \geq C\sqrt{n}$$

for a positive constant C. One can get a similar bound for $|\Pi_n(123)|^{1/n}$ when n is odd. In fact, one can analyze all the patterns considered in the previous section to prove the following result.

Theorem 4.1. *For the following patterns, the limit (4.1) is finite:*

$1/2/\dots/k$ *and all patterns with at most three elements except 123.*

For the following patterns, the limit (4.1) is infinite:

$12\dots k$ *for* $k \geq 3$ *and* $12\dots(k-1)/k$ *for* $k \geq 4$.

Any limit satisfies one of three possibilities: it exists and is finite, it exists and is infinite, or it does not exist. For a large class of partitions, the third possibility does not occur. Define $\sigma = C_1/\dots/C_k$ to be *reducible* if there is a j with $0 < j < n$ such that, possibly after reindexing the blocks, we have

$$C_1/\dots/C_i \vdash [j] \quad \text{and} \quad C_{i+1}/\dots/C_k \vdash [j+1, n].$$

Otherwise, σ is said to be *irreducible*. One can use Fekete's Lemma to prove the following.

Theorem 4.2. *If σ is irreducible then the limit (4.1) exists (and may be infinite).*

With this evidence, we make the following conjecture cum problem.

Conjecture 4.2. For every partition σ, the limit (4.1) exists. Characterize those patterns for which it is finite.

5 The second definition

Partitions can be viewed as certain sequences called restricted growth functions. A sequence of positive integers $p = a_1 a_2 \dots a_n$ is called a *restricted growth function (RGF) of length n* if

$$a_1 = 1 \text{ and for } i \geq 2 \text{ we have } a_i \leq 1 + \max\{a_1, \dots, a_{i-1}\}.$$

We let R_n denote the set of RGFs of length n and $R = \uplus_{n \geq 1} R_n$.

To see the connection with partitions, we assume from now on that the blocks of $\pi = B_1/B_2/\dots/B_l$ will always be indexed so that

$$\min B_1 < \min B_2 < \dots < \min B_l \tag{5.1}$$

Now given $\pi \vdash [n]$, we construct a sequence $p(\pi) = a_1 a_2 \dots a_n$ by letting $a_i = j$ iff $i \in B_j$. So, for example, $p(137/28/456/9) = 121333124$. It is easy to check that condition (5.1) forces $p(\pi)$ to be an RGF. Furthermore, this map is invertible, setting up a bijection $p : \Pi_n \longleftrightarrow R_n$. We can now use (1.1) to obtain the second definition of containment of a partition pattern.

Definition 5.1. Let s be an RGF in R_k, called the pattern. Then $p \in R_n$ *contains the pattern s*, written $s \preceq p$, if there is a subword $p' = a_{i_1} a_{i_2} \dots a_{i_k}$ of p such that

$$r_S(p') = s$$

where $S = \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$.

As before, we say that p *avoids* s if $s \not\preceq p$ and write $R_n(s)$ and $R(s)$ for those RGFs in R_n and R, respectively, which avoid s. One can now derive many interesting results about $R_n(s)$ and $R(s)$ similar to those about $\Pi_n(\sigma)$ and $\Pi(\sigma)$ either by working directly with RGFs or after reformulating these definitions in terms of partitions themselves. Furthermore, most questions about patterns in partitions remain unanswered. We hope the reader will be stimulated to look at some of them.

References

[1] M. Albert, M. Atkinson, C. Handley, D. Holton, and W. Stromquist, On packing densities of permutations, *Electron. J. Combin.* **9** (2002), R5, 20 pp.

[2] M. Bóna, The solution of a conjecture of Wilf and Stanley for all layered patters, *J. Combin. Theory Ser. A* **85** (1999), 96–104.

[3] M. Bóna, B. Sagan, and V. Vatter, Pattern frequency sequences and internal zeros *Adv. in Applied Math.* (special issue in memory of Rodica Simion), **28** (2002), 395–420.

[4] D. Foata, "La Série Génératrice Exponentielle dans les Problémes d'Enumération," Séminaire de Mathématique Supérieurs, No. 54, Presses de l'Université de Montréal, Montréal, 1974.

[5] T. Mansour and A. Vainshtein, Layered restrictions and Chebyshev polynomials, *Annals of Combin.* **5** (2001), 451–458.

[6] A. Price, "Packing densities of layered patterns," Ph.D. thesis, University of Pennsylvania, Philadelphia, PA, 1997.

[7] W. Stromquist, Packing layered posets into posets, manuscript.

Taming sets of permutations

Vincent R. Vatter[*]

Department of Mathematics
Rutgers University, Piscataway, NJ 08854
`vatter@math.rutgers.edu`

1 Introduction

It is an old and often rediscovered fact that there are infinite antichains of permutations with respect to the pattern containment ordering. See, for example, Pratt [4], Tarjan [6], and Speilman and Bóna [5]. Recall that we say that a partially ordered set is *well partially ordered (WPO)* if it contains neither an infinite properly decreasing sequence nor an infinite antichain, so these constructions show that the set of all finite permutations is not WPO. Nevertheless, Atkinson et al. [2] found several natural subsets of this poset that are WPO, for example, the set of permutations avoiding two different permutations of length three. We continue this investigation here.

First we need a few definitions. The *reduction* of a word w of distinct integers of length k is the k-permutation obtained by replacing the smallest element of w by 1, the second smallest element by 2, and so on. If $q \in S_k$, we say that the permutation $p \in S_n$ *contains a q pattern*, and write $q \le p$, if and only if there is a subsequence $1 \le i_1 < \cdots < i_k \le n$ so that $p(i_1)\dots p(i_k)$ reduces to q. Otherwise we say that p is q-avoiding (and write $q \not\le p$).

If Q is any set of permutations, we let $\mathcal{A}(Q)$ denote the set of finite permutations that avoid every member of Q. We also let $\mathrm{cl}(Q)$ denote the *closure* of Q, that is, the set of all permutations p such that there is a $q \in Q$ that contains p. We say that the set Q is *closed* (or that it is an *ideal* or a *down-set*) if $\mathrm{cl}(Q) = Q$.

If Q is any set of permutations, we define the *taming set of* Q, $\mathrm{tame}(Q)$, to be the set of permutations p such that $\mathcal{A}(Q \cup \{p\})$ is WPO. If $q \le p \in \mathrm{tame}(Q)$, then $\mathcal{A}(Q\cup\{q\}) \subseteq \mathcal{A}(Q\cup\{p\})$, so taming sets are closed. For example, Atkinson et al. [2] showed that $\mathrm{tame}(\emptyset) = \{1, 12, 21, 132, 213, 231, 312\}$ and they found $\mathrm{tame}(321)\cap S_4$. We will show later that $\mathrm{tame}(321) = \mathcal{A}(321, 2341, 3412, 4123)$, as an application of the much more general Theorem 2.4. In particular, this shows that taming sets may be infinite. Also note that by symmetry, this determines the taming sets of all permutations of length three.

Our arguments rely heavily on Higman's result that the set of finite words over a WPO set is WPO. More precisely, if $(X, \le)$ is a poset let X^* denote the set of all finite words with letters from X. We say that $a = a_1 \dots a_k$ is a subword of $b = b_1 \dots b_n$ (and write $a \le b$) if there is a subsequence $i_1 \le \cdots \le i_k$ such that $a_j \le b_{i_j}$ for all $j \in [k]$.

Theorem 1.1. *[3] If $(X, \le)$ is WPO then so is $(X^*, \le)$.*

[*]This work has been partially supported by an NSF VIGRE grant to the Rutgers University Department of Mathematics.

Our first application of this theorem is to direct sums of permutations. If $p \in S_m$ and $p' \in S_n$, we define $p \oplus p'$ to be the $(m+n)$-permutation given by

$$(p \oplus p')(i) = \begin{cases} p(i) & \text{if } 1 \le i \le m, \\ p'(i-m)+m & \text{if } m+1 \le i \le m+n. \end{cases}$$

If X is any set of permutations, set $\bigoplus^n X = \{p_1 \oplus \cdots \oplus p_n : p_1, \ldots, p_n \in X\}$, and let $\bigoplus X$ consist of all finite direct sums of elements from X, i.e., $\bigoplus X = \bigcup_{n \ge 0} \bigoplus^n X$. This set is WPO if X is WPO by Higman's Theorem 1.1.

2 Profile classes of $0/\pm 1$ matrices

Recall that the permutation matrix corresponding to $p \in S_n$, $M_p = (m_{i,j})_{i,j \in [n]}$, is given by

$$m_{i,j} = \begin{cases} 1 & \text{if } j = p(i), \\ 0 & \text{otherwise.} \end{cases}$$

If $M = (m_{i,j})$ is a $0/1$ matrix, we define the *support of* M, $\mathrm{supp}(M)$, to be the set of pairs (i,j) such that $m_{i,j} = 1$. If P and Q are both $0/1$ matrices, we say that P contains a Q pattern if there is a submatrix of P, say P', with $\mathrm{supp}(Q) \subseteq \mathrm{supp}(P')$ (note that here we have implicitly re-indexed the support of P', so we do not necessarily have $\mathrm{supp}(P') \subseteq \mathrm{supp}(P)$). We write $Q \le P$ when P contains a Q pattern. If q and p are permutations then $q \le p$ if and only if $M_q \le M_p$.

We define the *reduction* of a matrix M to be the matrix $\mathrm{red}(M)$ obtained from M by removing the all-zero columns and rows. Given a set of ordered pairs X let $\Delta(X)$ denote the smallest $0/1$ matrix with $\mathrm{supp}(\Delta(X)) = X$. If $\mathrm{red}(Q) = Q$ (for instance, if Q is a permutation matrix) then $Q \le P$ if and only if there is a set $X \subseteq \mathrm{supp}(P)$ with $\mathrm{red}(\Delta(X)) = Q$.

If $M = (m_{i,j})$ then we let $M_{I \times J} = (m_{i,j})_{i \in I, j \in J}$ and we let M^T denote the transpose of M. We define the *direct sum* of the matrices M_1 and M_2, written $M_1 \oplus M_2$, to be the matrix

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

This definition agrees with the one we made for permutations in the sense that $M_{p \oplus p'} = M_p \oplus M_{p'}$. If X is a set of $0/1$ matrices and $n \ge 0$, we let $\bigoplus^n X$ denote the set of all matrices of the form $M_1 \oplus \cdots \oplus M_n$ where $M_1, \ldots, M_n \in X$, and we set

$$\bigoplus X = \bigcup_{n \ge 0} \bigoplus^n X.$$

As with permutations, if X is a WPO set of permutation matrices (or, more generally, $0/1$ matrices) then Higman's Theorem 1.1 shows that $\bigoplus X$ is WPO.

We say that M is a *sub-permutation matrix* if there is a permutation matrix M' that contains an M pattern, or equivelantly, if $\mathrm{red}(M)$ is a permutation matrix. If M is a sub-permutation matrix and $\mathrm{supp}(M) = \{(i_1, j_1), \ldots, (i_\ell, j_\ell)\}$ with $1 \le i_1 < \cdots < i_\ell$, we say that M is *increasing* if $1 \le j_1 < \cdots < j_\ell$ and *decreasing* if $j_1 > \cdots > j_\ell \ge 1$.

Suppose that $M = (m_{i,j})$ is an $r \times s$ $0/\pm 1$ matrix and P is a permutation matrix. An *M-partition of* P is a pair (I, J) of multisets $I = \{1 = i_1 \le \cdots \le i_{r+1} = n+1\}$ and $J = \{1 = j_1 \le \cdots \le j_{s+1} = n+1\}$ such that

(i) if $m_{k,\ell} = 0$ then $P_{[i_k, i_{k+1}) \times [j_\ell, j_{\ell+1})} = 0$,

(ii) if $m_{k,\ell} = 1$ then $P_{[i_k, i_{k+1}) \times [j_\ell, j_{\ell+1})}$ is increasing,

(iii) if $m_{k,\ell} = -1$ then $P_{[i_k, i_{k+1}) \times [j_\ell, j_{\ell+1})}$ is decreasing.

For any $0/\pm 1$ matrix M we define the *profile class of* M, $\mathrm{Prof}(M)$, to be the set of all permutation matrices that admit an M-partition, and we let $\mathrm{Part}(M)$ consist of the set of triples (P, I, J) where $P \in \mathrm{Prof}(M)$ and (I, J) is an M-partition of P.

The profile classes of $0/\pm 1$ matrices defined here generalize both the profile classes of permutations used by Atkinson [1] and the "generalized W's," used by Atkinson et al. [2]. Unlike those two constructions, it is not true that all of these classes are WPO. For example, consider the infinite sequence of permutations $W = \{w_1, w_2, \dots\}$ given by

$$\begin{array}{rcl}
w_1 &=& 8,1 \mid 5,3,6,7,9,4 \mid \mid 10,11,2, \\
w_2 &=& 12,1,10,3 \mid 7,5,8,9,11,6 \mid 13,4 \mid 14,15,2, \\
& \dots & \\
w_k &=& 4k+4, 1, 4k+2, 3, \dots, 2k+6, 2k-1 \mid \\
&& 2k+3, 2k+1, 2k+4, 2k+5, 2k+7, 2k+2 \mid \\
&& 2k+9, 2k, 2k+11, 2k-2, \dots, 4k+5, 4 \mid \\
&& 4k+6, 4k+7, 2,
\end{array}$$

where the vertical bars indicate that w_k consists of four different parts, of which the first part is the interleaving of $4k+4, 4k+2, \dots, 2k+6$ with $1, 3, \dots, 2k-1$, the second part consists of just six terms, the third part is the interleaving of $2k+9, 2k+11, \dots, 4k+5$ with $2k, 2k-2, \dots, 4$, and the fourth part has only 3 terms. Atkinson et al. [2] showed that W is an antichain. Furthermore, each M_{w_k} has a $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$-partition: $(\{1, 2k+3, 4k+8\}, \{1, 2k+3, 4k+8\})$. For example,

$$M_{w_2} = \left(\begin{array}{cccccc|ccccccccc}
&&&&&&&&&&&1&&& \\
1&&&&&&&&&&&&&& \\
&&&&&&&&&1&&&&& \\
&&1&&&&&&&&&&&& \\
&&&&&&1&&&&&&&& \\
&&&&1&&&&&&&&&& \\
\hline
&&&&&&&1&&&&&&& \\
&&&&&&&&1&&&&&& \\
&&&&&&&&&&1&&&& \\
&&&&&1&&&&&&&&& \\
&&&&&&&&&&&&1&& \\
&&&1&&&&&&&&&&& \\
&&&&&&&&&&&&&1& \\
&&&&&&&&&&&&&&1 \\
&1&&&&&&&&&&&&&
\end{array}\right) \in \mathrm{Prof}\left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right).$$

Therefore $\mathrm{Prof}\left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right)$ is not WPO under the pattern containment ordering.

Suppose that M is an $r \times s$ $0/\pm 1$ matrix with $(P, I, J), (Q, I', J') \in \mathrm{Part}(M)$ where $I = \{i_1 \le \cdots \le i_{r+1}\}$, $J = \{j_1 \le \cdots \le j_{s+1}\}$, $I' = \{i'_1 \le \cdots \le i'_{r+1}\}$, and $J' = \{j'_1 \le \cdots \le j'_{s+1}\}$. Then we write $(Q, I', J') \preceq (P, I, J)$ if there is a set $X \subseteq \mathrm{supp}(P)$ such that $\mathrm{red}(\Delta(X)) = Q$ and for all $k \in [r]$ and $\ell \in [s]$,

$$|X \cap ([i_k, i_{k+1}) \times [j_\ell, j_{\ell+1}))| = |\mathrm{supp}(Q) \cap ([i'_k, i'_{k+1}) \times [j'_\ell, j'_{\ell+1}))|.$$

The poset we are really interested in, $(\mathrm{Prof}(M), \le)$, is a homomorphic image of $(\mathrm{Part}(M), \preceq)$. Consequently, if for some M we can show that $(\mathrm{Part}(M), \preceq)$ is

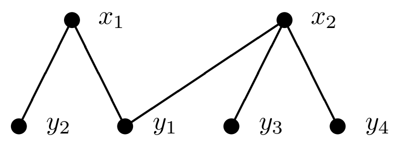


Figure 1: $G\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}\right)$

WPO, then we may conclude that $(\mathrm{Prof}(M), \leq)$ is WPO. The following proposition shows that we need only examine the cases where M is a 0/1 matrix.

Proposition 2.1. *Let $M = (m_{i,j})_{i\in[r],j\in[s]}$ and $M' = (m'_{i,j})_{i\in[r],j\in[s]}$ be $0/\pm 1$ matrices with $|m_{i,j}| = |m'_{i,j}|$ for all $i \in [r]$, $j \in [s]$. Then $(Part(M), \preceq)$ is WPO if and only if $(Part(M'), \preceq)$ is WPO.*

The next two propositions also follow easily from the definitions.

Proposition 2.2. *Let M be a $0/\pm 1$ matrix. Then $(Part(M^T), \preceq)$ is WPO if and only if $(Part(M), \preceq)$ is WPO.*

Proposition 2.3. *Let M be a $0/\pm 1$ matrix and suppose that M' can be obtained by permuting the rows and columns of M. Then $(Part(M), \preceq)$ is WPO if and only if $(Part(M'), \preceq)$ is WPO.*

Note that the analogue of Proposition 2.2 for the poset $(\mathrm{Prof}(M), \leq)$ is true, whereas the analogues of Propositions 2.1 and 2.3 are not obvious.

If $M = (m_{i,j})$ is an $r\times s$ $0/\pm1$ matrix we define the *bipartite graph* of M, $G(M)$, to be the graph with vertices $\{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_s\}$ and edges $\{(x_i, y_j) : |m_{i,j}| = 1\}$. Figure 1 shows an example. Propositions 2.1, 2.2, and 2.3 suggest that whether or not $(\mathrm{Part}(M), \preceq)$ is WPO depends only on the isomorphism class of $G(M)$. The main result of this section, below, characterizes these graphs.

Theorem 2.4. *Let M be a $0/\pm 1$ matrix. Then $(Part(M), \preceq)$ is WPO if and only if $G(M)$ is a forest.*

It seems natural to expect that Theorem 2.4 would remain true if $(\mathrm{Part}(M), \preceq)$ were replaced by $(\mathrm{Prof}(M), \leq)$. One direction is already done; Theorem 2.4 shows that $(\mathrm{Prof}(M), \leq)$ is WPO whenever $G(M)$ is a forest. It only remains to show that $(\mathrm{Prof}(M), \leq)$ is not WPO if $G(M)$ contains a cycle.

Conjecture 2.5. *Let M be a $0/\pm 1$ matrix. Then $(Prof(M), \leq)$ is WPO if and only if $G(M)$ is a forest.*

3 The taming set of 321

Theorem 2.4 is used to prove the following theorem.

Theorem 3.1. *The taming set of* 321 *is* $\mathcal{A}(321, 2341, 3412, 4123)$.

Like Theorem 2.4, the proof of Theorem 3.1 is rather technical, but we will try to shed some light on it with a few comments. First, to show that

$$\mathrm{tame}(321) \subseteq \mathcal{A}(321, 2341, 3412, 4123),$$

we need only exhibit various infinite antichains, and this work has already been done by Atkinson et al. [2].

The other direction is more demanding. We begin by finding a more convenient description for $\mathcal{A}(321, 2341, 3412, 4123)$. For all $k \geq 1$, let b_k denote the permutation obtained from the reduction of the first k terms of the infinite sequence

$$4, 1, 6, 3, 8, 5, \ldots,$$

and let $B = \{b_1, b_2, \ldots\}$. We show that

$$\mathcal{A}(321, 2341, 3412, 4123) = \mathrm{cl}\left(\bigoplus\left(B \cup B^{-1}\right)\right).$$

Because taming sets are closed, it suffices to show that

$$\bigoplus\left(B \cup B^{-1}\right) \subseteq \mathrm{tame}(321).$$

Furthermore, B and B^{-1} form chains in the pattern containment ordering, so it suffices to show that

$$\bigoplus^{j}\{b_k, b_k^{-1}\} \subseteq \mathrm{tame}(321)$$

for all $j, k \geq 1$. We prove, by double induction on j and k, that for all $q \in \bigoplus^j\{b_k, b_k^{-1}\}$, there is a matrix $M(q)$ with $G(M(q))$ a forest so that if $p \in \mathcal{A}(321, q)$ then $M_p \in \bigoplus \mathrm{Prof}(M(q))$. We are then done by Higman's Theorem 1.1 and Theorem 2.4.

Acknowledgment. I am grateful to Doron Zeilberger for his enthusiastic and gracious support.

References

[1] M. D. Atkinson, Restricted permutations, *Discrete Math.* **195** (1999) 27-38.

[2] M. D. Atkinson, M. M. Murphy, and M. Ruskuc, Partially well-ordered closed sets of permutations, preprint.

[3] G. Higman, Ordering by divisibility in abstract algebra, *Proc. London Math. Soc.* **2** (1952), 326-336.

[4] V. R. Pratt, Computing permutations with double-ended queues, parallel stacks and parallel queues, *Proc. ACM Symp. Theory of Computing* **5** (1973) 268-277.

[5] D. Spielman, M. Bóna, An infinite antichain of permutations, *Electronic J. Combinatorics* **7** (2000), #N2.

[6] R. E. Tarjan, Sorting using networks of queues and stacks, *J. of the ACM* **19** (1972), 341-346.

Is there always a *large* Wilf class, and why isn't it larger?

Julian West

We are quite good at showing that two sets of permutation patterns are equally restrictive. We are less good at showing that two sets of patterns are not equally restrictive, the usual approach being brute-force enumeration. In particular, it seems credible that we know about all the cases of Wilf-equivalence for single patterns. But we don't know how to show that there are no further instances. In this talk, we will survey some of the relevant literature, highlighting suggested approaches for proving inequivalence.

Longest increasing subsequences in pattern-restricted permutations

Herb Wilf

Recent breakthroughs by Baik, Deift and Johansson have resulted in the discovery of the complete limiting distribution function of the length of the longest increasing subsequence in a random permutation. The same questions can be asked in pattern-restricted classes of permutations. For example, what, asymptotically, is the length of the longest increasing subsequence in a random (132)-avoiding permutation? What is the complete limiting distribution function? Some recent results of Chow and West, and of Reifegerste, when viewed in the light of these breakthroughs, allow us to find the answers to some of these questions and to raise many others. This is joint work with Adolf Hildebrand, of the University of Illinois.