Bubblesort and permutations

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Outline of talk

1. Permutations and Pattern classes: basic concepts
2. One pass bubblesort
3. Results and applications

Permutations

- A permutation of length n is an arrangement of $1, 2, \ldots, n$ (one-line notation, not cycle notation)

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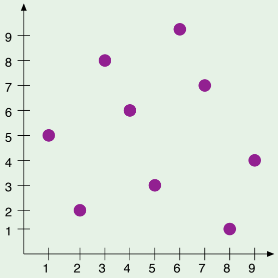
Example

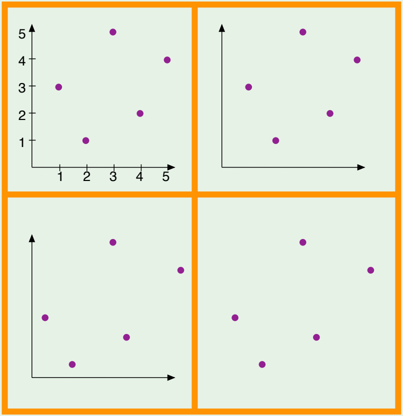
The graph of 528639714



The cardinal sin: unlabeled axes

Example

Increasingly sloppy graphs of 31524



Subpermutations

Example

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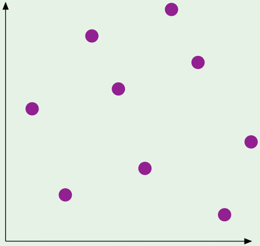
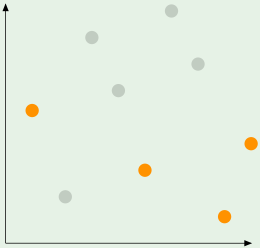


Subpermutations

Example

The graph of 528639714 and subpermutation 4213



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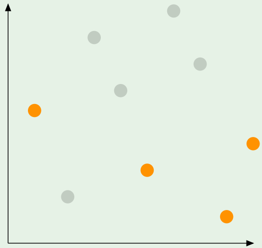


4213 is a subpermutation of 528639714.

Notation

$4213 \preceq 528639714$ *and* $54321 \not\preceq 528639714$

The $\preceq$ relation is a partial order on the set of all permutations

Pattern classes

Definition

A *pattern class* is a set of permutations closed under taking subpermutations (down-set in the partial order)

Every pattern class $\mathcal{X}$ can be defined by a set of avoided permutations.

Notation

$\mathrm{Av}(B)$*: the pattern class defined by avoiding permutations in the set* B*.*

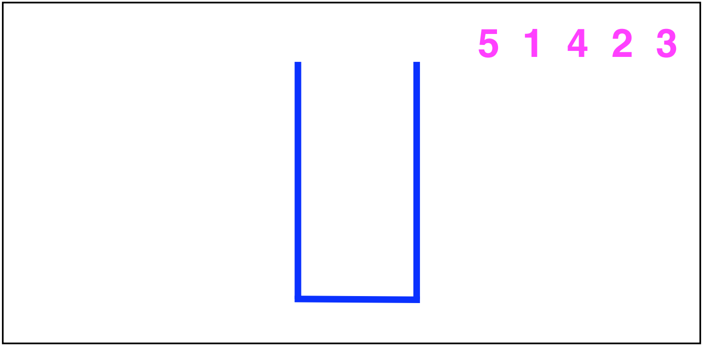
Main issues

- *Given a pattern class realise it in the form* $\mathrm{Av}(B)$ *for some set* B
- *Give structural description of permutations in some given* $\mathrm{Av}(B)$
- *Count them of each length*

Example 1: all identity permutations

The set of all identity permutations $\{1, 12, 123, 1234, \ldots\}$ forms the pattern class $\mathrm{Av}(21)$. It has 1 permutation of every length *n*.

Example 2: The one pass Stacksort operator



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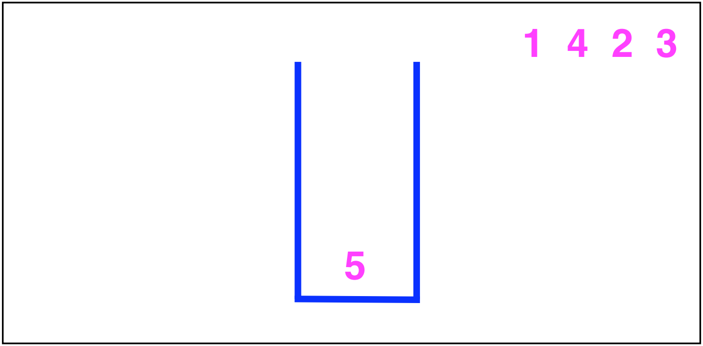

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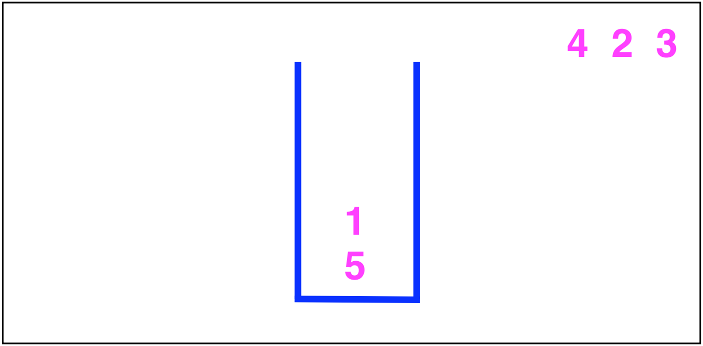

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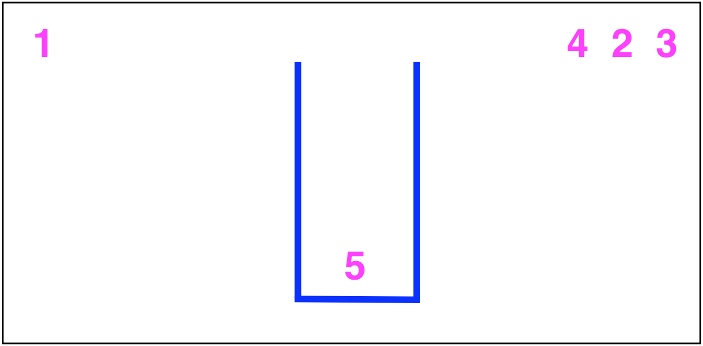

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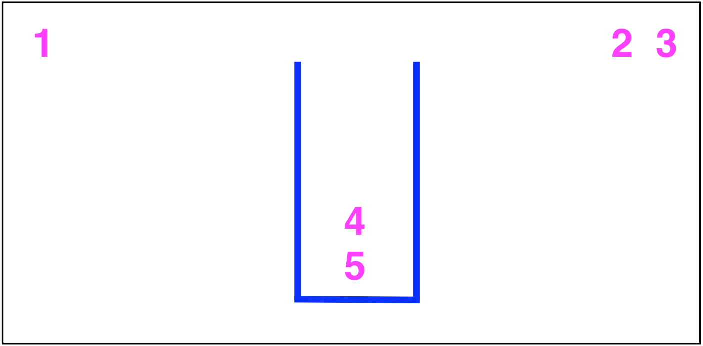

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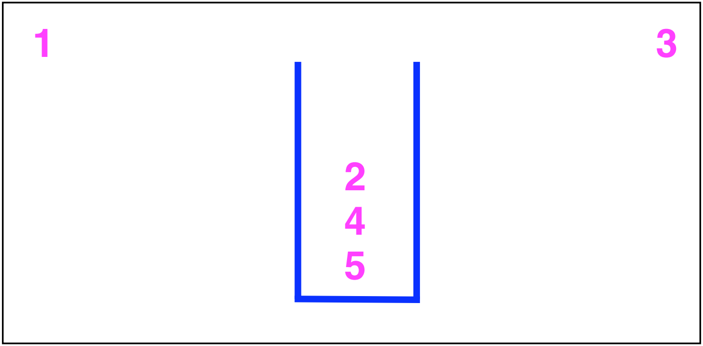

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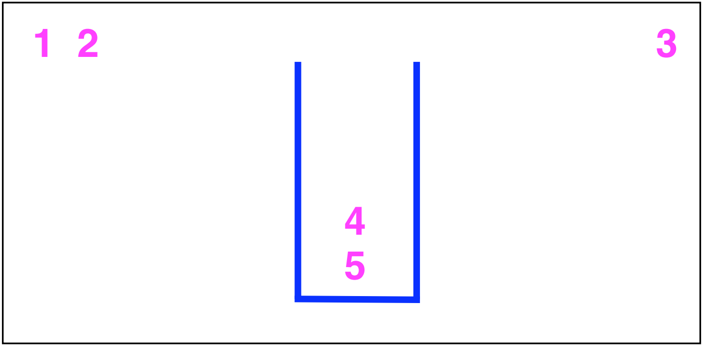

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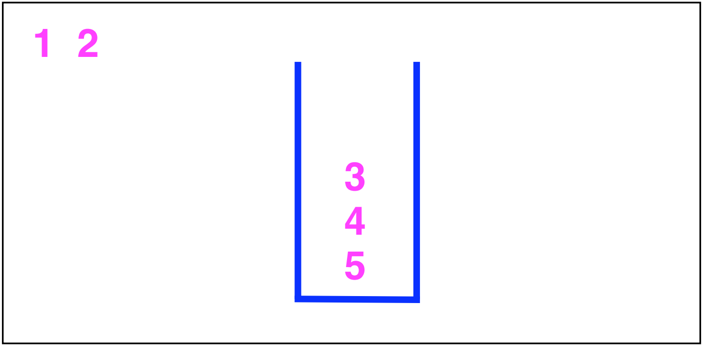

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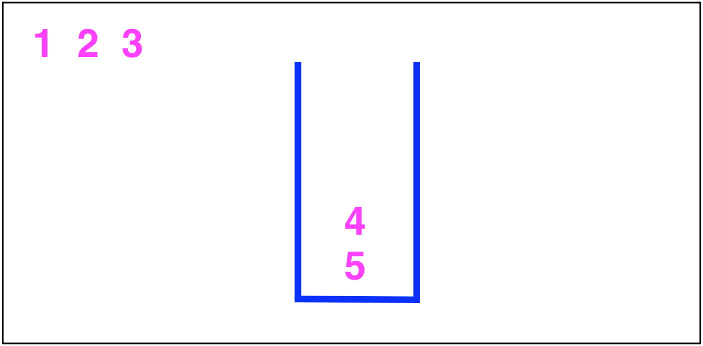

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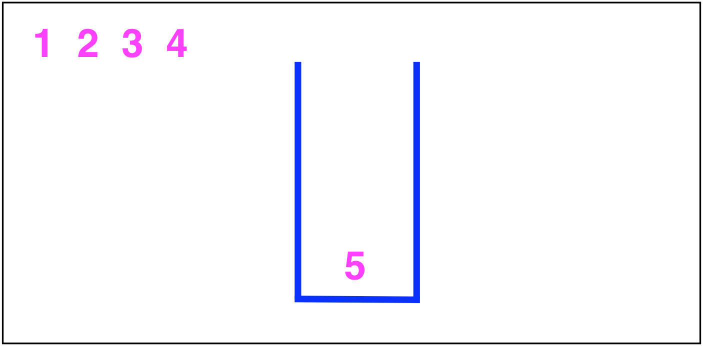

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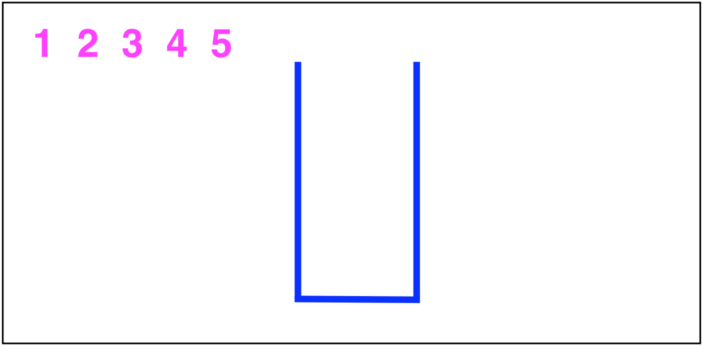

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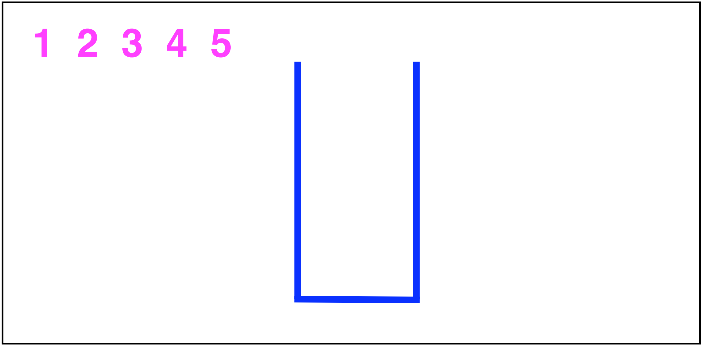

The stack-sortable permutations form the pattern class $\mathrm{Av}(231)$. It has $\frac{\binom{2n}{n}}{n+1}$ permutations of length n

$\mathrm{Av}(4312, 2143)$ – movie of length 13 permutations

The Bubblesort operator B

1. Bubblesort is an inefficient sorting algorithm that proceeds in a number of passes.
2. Each pass is a left to right scan

4	1	2	7	6	3	9	8	5

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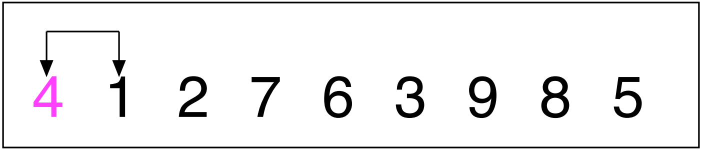

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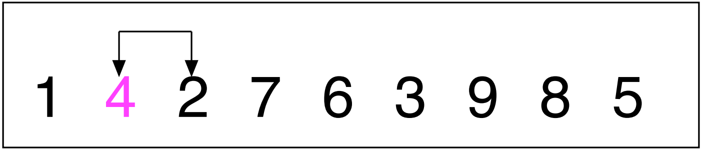

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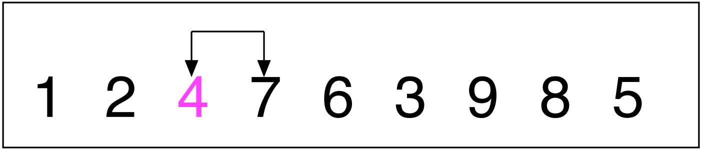

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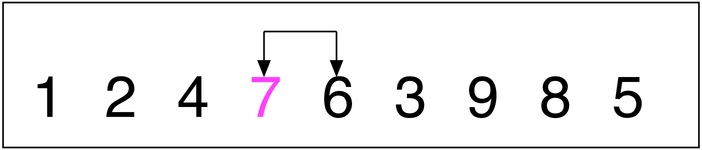

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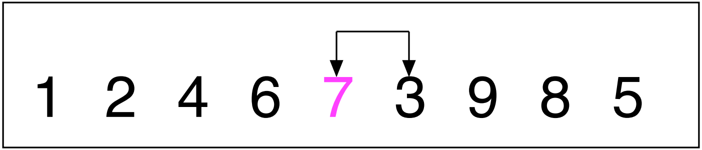

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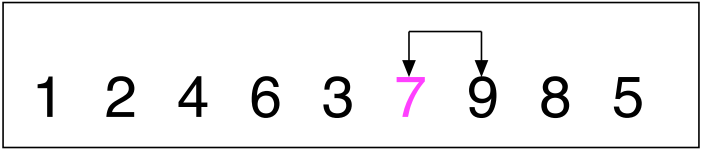

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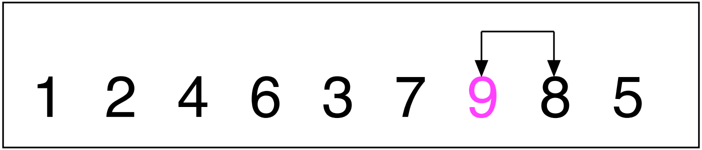

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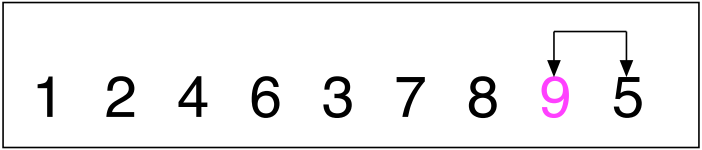

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The effect $\sigma \to B(\sigma)$ of one pass is given by

$$B(\epsilon) = \epsilon \text{ and}$$

$$B(\alpha n \beta) = B(\alpha)\beta n$$

B sorts a pattern class

1. In general we need n passes to sort a permutation of length n
2. But B (applied once) sorts a small number of permutations
3. The set sortable by B is $\mathrm{Av}(231, 321)$

Theorem

$$B^{-1}(\mathrm{Av}(21)) = \mathrm{Av}(231, 321)$$

A general question

Formally generalising the statement of this theorem:

Question

1. *For an arbitrary pattern class* $\mathrm{Av}(\Pi)$ *what is* $B^{-1}(\mathrm{Av}(\Pi))$*?*
2. *Is it even a pattern class? If so, which pattern class is it?*

An answer would enable the study of composing B with other sorting operators.

Left to right maxima

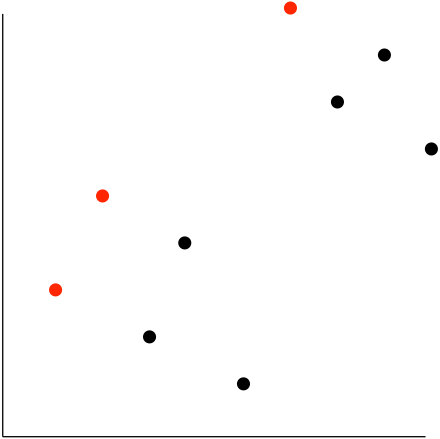

Figure: 352419786 has 3 left to right maxima

Theorems

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Theorem

If Π *consists of good permutations then* $B^{-1}(\mathrm{Av}(\Pi))$ *is a pattern class. Moreover we have* $B^{-1}(\mathrm{Av}(\Pi)) = \mathrm{Av}(\Sigma)$ *for some set* Σ *that is explicitly computable in terms of* Π *– but this margin...*

Example application

Question

How many permutations of length n can be sorted by one pass of Bubblesort followed by one pass of Stacksort?

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- The explicit forms of the theorem above tells us that $B^{-1}(\mathrm{Av}(231)) = \mathrm{Av}(3241, 2341, 4231, 2431)$
- Now a calculation that is quite commonplace in this subject shows that this set has $\binom{2n-2}{n-1}$ permutations of each length n.

Lots more questions

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2. Any answers then allow us to investigate the permutations sortable by combinations of phases from different sorting algorithms

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Thank you for listening