Candy Nim

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- Definition of the game
- Basic results
- Three heap candy Nim

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- He could decide to collect beans, or better yet, candies.

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- Subject to the above, both players play to maximize the number of candies which they collect.

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- $G + H$ denotes the concatenation of G and H.
- $v(G)$ denotes the *value* of G in candy Nim, that is, the difference between the number of candies collected by the first player, and the number collected by the second player under optimal play.

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In fact, except in positions where every pile size occurs an even number of times, the first player can guarantee a positive outcome by always taking all of the largest pile.

Value is sub-additive

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A variation on strategy stealing. For the right hand inequality, the second player plays separately in G and H. For the left, the first player avoids playing on H unless the second player answers a move in G with one in H. In that case he takes the second player's move there. □

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- $v(1,2,3+1,2,3) = 0 = v(1,2,3) - v(1,2,3)$.
- The proposition implies that, in general, we can delete pairs of equal sized heaps when computing a value.

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- What is the value?
- Where can the first player move effectively?

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- By moving from $1, 2k, 2k + 1$ to $1, 2k, 2k - 2$, you get a 3 to 1 advantage when the second player makes her reply to $1, 2k - 1, 2k - 2$.

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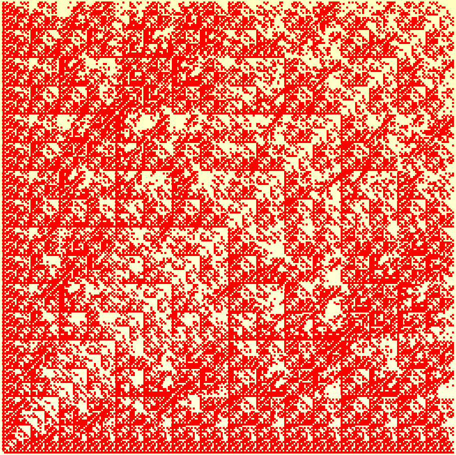
- By moving from $1, 2k, 2k+1$ to $1, 2k, 2k-2$, you get a 3 to 1 advantage when the second player makes her reply to $1, 2k-1, 2k-2$.
- This is easily seen to be optimal and so, inductively:

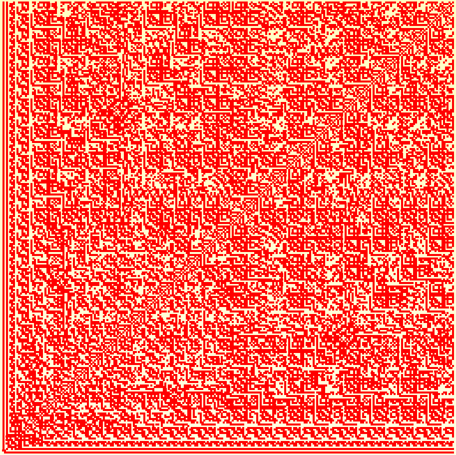
$$v(1, 2k, 2k+1) = 2k.$$

Move in biggest heap?



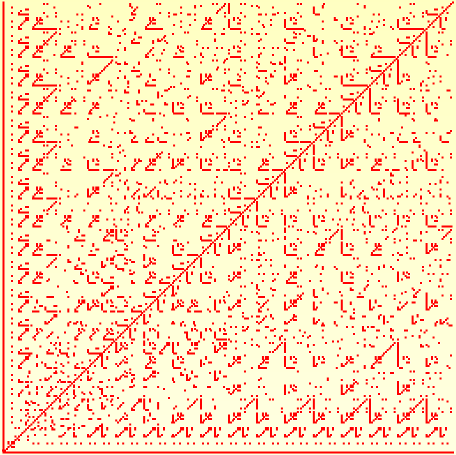
- Red means that there is a good move in the biggest heap.
- Plot is for $a, b, a \oplus b$ with $0 \leq a, b \leq 255$.

Unique good move?



- Red means that there is *not* a unique good move.

Fair first exchange?



- Red means that the first player cannot gain candies on the first exchange.

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Thank you.