Cyclic Increasing Sequences

Michael Albert[1] Mike Atkinson[1] Doron Nussbaum[2] Jörg Sack[2] Nicola Santoro[2]

[1]Department of Computer Science, University of Otago

[2]School of Computer Science, Carleton University

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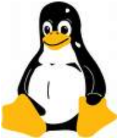

Outline of talk

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- What is typical?
- (Ulam, 1960's) Given a (uniformly) random arrangement of $1, 2, \ldots, n$ what is the expected length μ_n of its longest increasing subsequence?

The expected length μ_n of the LIS: history

- Ulam conjectured that

$$K_{LIS} = \lim_{n\to\infty} \frac{\mu_n}{\sqrt{n}}$$

exists; i.e. $\mu_n \sim K_{LIS}\sqrt{n}$.

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$$\begin{aligned} \mu_n &= 2n^{1/2} - \gamma n^{1/6} + o(n^{1/6}) \\ \sigma_n &= \delta n^{1/6} + o(n^{1/6}) \end{aligned}$$

where $\gamma = 1.711$ and $\delta = 0.902$.

The LIS algorithm: Schensted (1961)

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What does "best" mean?
Most easily extended – i.e. having smallest last term.
Often just need the final term of a best IS.

Example: LIS of 25, 30, 18, 20, 56, 21, 45, 17, 40, 15, 43, 16

	25	30	18	20	56	21	45	17	40	15	43	16
1	25											
2												
3												
4												
5												

Example: LIS of 25, 30, 18, 20, 56, 21, 45, 17, 40, 15, 43, 16

	25	30	18	20	56	21	45	17	40	15	43	16
1	25	25										
2		30										
3												
4												
5												

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	25	30	18	20	56	21	45	17	40	15	43	16
1	25	25	18									
2		30	30									
3												
4												
5												

Example: LIS of 25, 30, 18, 20, 56, 21, 45, 17, 40, 15, 43, 16

	25	30	18	20	56	21	45	17	40	15	43	16
1	25	25	18	18								
2		30	30	20								
3												
4												
5												

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	25	30	18	20	56	21	45	17	40	15	43	16
1	25	25	18	18	18							
2		30	30	20	20							
3					56							
4												
5												

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	25	30	18	20	56	21	45	17	40	15	43	16
1	25	25	18	18	18	18						
2		30	30	20	20	20						
3					56	21						
4												
5												

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1	25	25	18	18	18	18	18					
2		30	30	20	20	20	20					
3					56	21	21					
4							45					
5												

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1	25	25	18	18	18	18	18	17				
2		30	30	20	20	20	20	20				
3					56	21	21	21				
4							45	45				
5												

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1	25	25	18	18	18	18	18	17	17			
2		30	30	20	20	20	20	20	20			
3					56	21	21	21	21			
4							45	45	40			
5												

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2		30	30	20	20	20	20	20	20	20		
3					56	21	21	21	21	21		
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2		30	30	20	20	20	20	20	20	20	20	
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4							45	45	40	40	40	
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4							45	45	40	40	40	40
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2		30	30	20	20	20	20	20	20	20	20	16
3					56	21	21	21	21	21	21	21
4							45	45	40	40	40	40
5											43	43

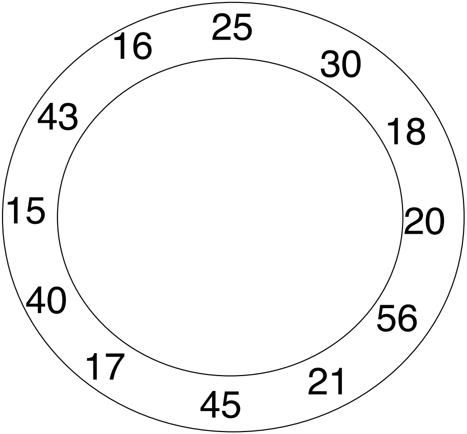
The LIS has length 5 and is $18, 20, 21, 40, 43$

Analysis

- Only store the last column
- New last column obtained from old by binary search and insertion
- Some back pointers to actually construct the result
- Total time $O(n \log n)$

The Longest Increasing Circular Subsequence (LICS)

$25, 30, 18, 20, 56, 21, 45, 17, 40, 15, 43, 16$ in a circle.



The LICS is not $18, 20, 21, 40, 43$ but is $15, 16, 18, 20, 21, 45$

LICS versus LIS

If π is a permutation then

- $|LIS(\pi)| \leq |LICS(\pi)| \leq 2|LIS(\pi)|$
- Upper bound attained for $\pi = k+1, k+2, \ldots, 2k, 1, 2, \ldots, k$
- Lower bound attained for $\pi = 1, 2, \ldots, n$

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- We could run the LIS algorithm on the n different ways of regarding the circular sequence as a linear sequence.

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- Can we do better?

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Recent applications of LIS detection algorithms in matching gene sequences, and some bacteria have apparently circular sequences; special case of general permutation pattern class problems, guff, guff, guff....

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"Because it's there"

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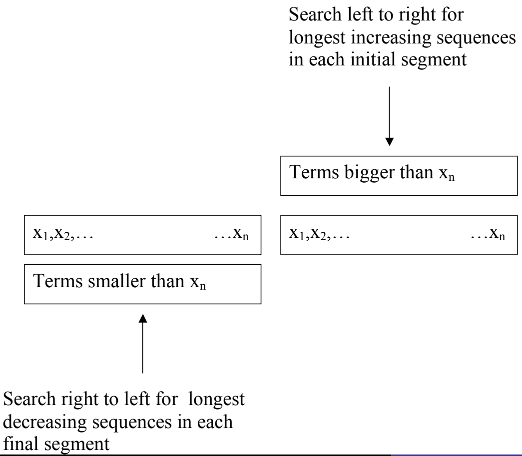
Passing through

Lemma

If $t_1, \ldots, t_m$ is the final column of the LIS algorithm then there is an LICS that contains at least one of these terms.

There and back again

We can find the LICS in time $O(n \log n)$ if we know any term that it contains.



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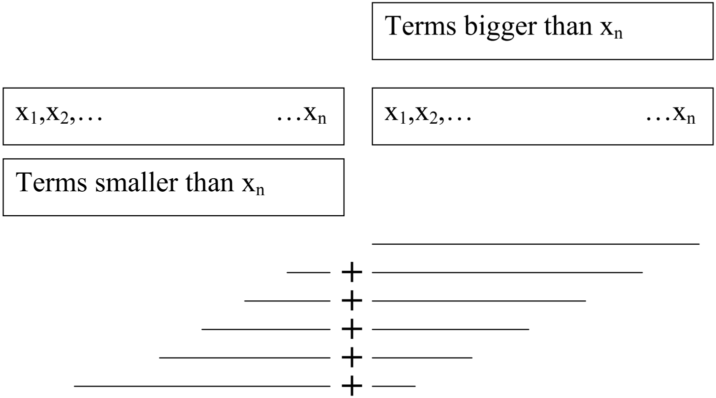

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- Return the longest of these

The execution time

Theorem

The expected execution time of the LICS algorithm is $O(n\sqrt{n}\log n)$.

Proof.

The expected value of m is $2\sqrt{n}$. There are $2m+1$ applications of the LIS algorithm each taking time $O(n\log n)$. $\square$

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- This is a practical algorithm

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  3. Make random guesses for a term s in $LICS(\pi)$; run "There and back" on such terms.
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- If we make more than $-\sqrt{n}\ln(\epsilon)/3$ guesses the probability that *none* of them lie in the LICS is less than ϵ.

Lies, damned lies and statistics

10000 random trials gave estimates μ_n-est. for μ_n for various n.

n	10000	12000	14000	16000	18000	20000
μ_n-est.	200.1	219.3	237.0	253.4	268.8	283.4

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$2\sqrt{n}$	200.0	219.1	236.7	253.0	268.3	282.8

It appears that μ_n is much closer to $2\sqrt{n}$ than it was in the linear case. But all we can prove is...

The mean of the LICS

Theorem

$$\lim_{n\to\infty} \frac{\mu_n}{\sqrt{n}} = 2.$$