Aspects of Separability

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Outline

- Separable permutations
- Subclasses are small
- Six degrees of separation, not!
- An example

Separable permutations

The class $\mathcal{S}$ of *separable permutations* consists of those that do not contain either of the patterns 2413 or 3142.

A more illuminating definition is that $\mathcal{S}$ is the closure of the singleton permutation under the operations $\oplus$ and $\ominus$, where:

$$\alpha \oplus \beta = 12[\alpha, \beta] = \begin{array}{c|c} & \beta \\ \hline \alpha & \end{array}$$

$$\alpha \ominus \beta = 21[\alpha, \beta] = \begin{array}{c|c} \alpha & \\ \hline & \beta \end{array}$$

The master equations

$$\mathcal{S} = 1 \cup 12[\mathcal{S}, \mathcal{S}] \cup 21[\mathcal{S}, \mathcal{S}].$$

This is no good for computing generating functions, since the decomposition may not be unique. Let $\mathcal{S}^+$ and $\mathcal{S}^-$ be the *plus indecomposable* and *minus indecomposable* elements of $\mathcal{S}$ respectively. Then:

$$\begin{aligned} \mathcal{S} &= 1 \cup 12[\mathcal{S}^+, \mathcal{S}] \cup 21[\mathcal{S}^-, \mathcal{S}] \\ \mathcal{S}^+ &= 1 \cup 21[\mathcal{S}^-, \mathcal{S}] \\ \mathcal{S}^- &= 1 \cup 12[\mathcal{S}^+, \mathcal{S}] \end{aligned}$$

and this does directly produce a system of equations for the generating functions s, s^+ and s^-.

Details

The separable permutations are enumerated by the large Schröder numbers:

$$\begin{aligned} s &= \frac{1 - t - \sqrt{1 - 6t + t^2}}{2t} \\ &= t + 2t^2 + 6t^3 + 22t^4 + 90t^5 + 394t^6 + 1806t^7 + 8558t^8 + \cdots \end{aligned}$$

The radius of convergence of s is $r = 3 - 2\sqrt{2}$ and hence:

$$\lim_{n \to \infty} |\mathcal{S}_n|^{1/n} = \frac{1}{3 - 2\sqrt{2}} = 3 + 2\sqrt{2}.$$

Growth rate criticality

A class, $\mathcal{C}$, of permutations is said to be *growth rate critical* if, for any proper subclass $\mathcal{D} \subset \mathcal{C}$:

$$\limsup_{n\to\infty} |\mathcal{D}_n|^{1/n} < \limsup_{n\to\infty} |\mathcal{C}_n|^{1/n}.$$

A few sufficient conditions for growth rate criticality are known, but they do not apply to $\mathcal{S}$. So:

Question

Is $\mathcal{S}$ growth rate critical?

Yes

Consider subclasses $\mathcal{S}_\pi$ defined by a single additional pattern restriction. Argue inductively on $|\pi|$. In the inductive step, suppose without loss of generality that $\pi = \alpha \ominus \beta$ where α is minus indecomposable. Then define a superset $\mathcal{Q}_\pi$ of $\mathcal{S}_\pi$

$$\begin{aligned}
\mathcal{Q}_\pi &= 1 \cup 12[\mathcal{Q}_\pi^+, \mathcal{Q}_\pi] \cup 21[\mathcal{S}_\alpha^-, \mathcal{Q}_\pi] \cup 21[\mathcal{Q}_\pi^- \setminus \mathcal{S}_\alpha^-, \mathcal{S}_\beta] \\
\mathcal{Q}_\pi^+ &= 1 \cup 21[\mathcal{S}_\alpha^-, \mathcal{Q}_\pi] \cup 21[\mathcal{Q}_\pi^- \setminus \mathcal{S}_\alpha^-, \mathcal{S}_\beta] \\
\mathcal{Q}_\pi^- &= 1 \cup 12[\mathcal{Q}_\pi^+, \mathcal{Q}_\pi].
\end{aligned}$$

Now use the fact that the generating functions a and b of $\mathcal{S}_\alpha^-$ and $\mathcal{S}_\beta$ have radius of convergence larger than r, as well as $a, b < s$ to establish that the solution of this system has its smallest positive singularity at some $r' > r$.

Not many degrees of separation

Every proper subclass $\mathcal{C}$ of $\mathcal{S}$ has a generating function which is algebraic over $\mathbb{Q}(t)$. A *strong subclass* of a class C is one each of whose basis elements is involved in some basis element of $\mathcal{C}$.

Theorem

Let $\mathcal{C}$ be a proper subclass of $\mathcal{S}$. If the basis of $\mathcal{C}$ contains both plus and minus decomposable elements, then its generating function is rational over $\mathbb{Q}(t)$ and the generating functions of all its strong subclasses. Otherwise, it is of degree one or two over this field.

Corollary

The degree over $\mathbb{Q}(t)$ of the generating function of any subclass of $\mathcal{S}$ is a power of 2.

But as many as can be expected

Let $(\pi_n)_{n\geq 1}$ be the sequence of permutations:

$$132, 4132, 15243, 615243, 1726354, \ldots$$

that is, $\pi_1 = 132$ and

$$\pi_{n+1} = \begin{cases} 1 \oplus \pi_n & n \text{ even} \\ 1 \ominus \pi_n & n \text{ odd.} \end{cases}$$

Define $\mathcal{C}_n$ to consist of those permutations in $\mathcal{S}$ that also avoid π_n, and let $c_n(t)$ be its generating function.

Proposition

The degree over $\mathbb{Q}(t)$ of c_n is precisely 2^n.

Proof (sketch i)

The result is *obvious* by substituting in the master equation. But ...
We must ensure that there's no accidental collapse – i.e. that the quadratic which c_n satisfies over $\mathbb{Q}(t, c_{n-1})$ is truly irreducible. It turns out that:

$$c_{n-1} = \frac{c_n - c_n t - t}{c_n^2 + c_n + t}$$

and this gives a recurrence for a polynomial satisfied by c_n namely:

$$\left(x^2 + x + t\right)^{d_{n-1}} P_{n-1}\left(\frac{x - xt - t}{x^2 + x + t}\right)$$

where P_{n-1} is the irreducible polynomial satisfied by c_{n-1} and d_{n-1} its degree.

Proof (sketch ii)

If, when we formally substitute $t = 1$ in this recurrence we get a sequence of irreducible polynomials over $\mathbb{Q}$, we'd be done. And of course we do.

To see *that* we further reduce to work over $GF(2)$ rather than $\mathbb{Q}$ and consider a sequence of elements in some algebraic closure defined by: $\alpha_3^2 + \alpha_3 + 1 = 0$, and thereafter

$$\alpha_n^2 + \alpha_n + 1 = \frac{1}{\alpha_{n-1}}.$$

A bit of algebraic sleight of hand gives an infinite descent if α_n is rational in α_{n-1}.

Summary and questions

- The strong structure satisfied by $\mathcal{S}$ imposes significant restrictions on the form of the generating functions of its subclasses.
- To some extent, parallel methods apply for any class containing only finitely many simples.
- It would be reasonable to ask, given a permutation $\pi \in \mathcal{S}$, what is the degree of the generating function of $\mathcal{S}_\pi$? The last result suggests that this may well be related to the number of $\oplus$-$\ominus$ alternations in the description of π.