Simple permutations and substitution closures

Mike Atkinson

Department of Computer Science, University of Otago

PP2007, St Andrews, June 2007

Outline of talk

1. Background: pattern classes and simple permutations
   - Terminology
   - Skeletons
2. Substitution closed pattern classes
   - Generating functions
   - Growth rates
3. Counting simple permutations
4. Principal classes
   - Main questions
   - Finite types
   - Infinite types
   - A hint at the proofs

Terminology

- Subpermutation: 3142 is a subpermutation of 5624713

Terminology

- Subpermutation: 3142 is a subpermutation of 5624713
- Pattern class: set of permutations closed under taking subpermutations.

Terminology

- Subpermutation: 3142 is a subpermutation of 5624713
- Pattern class: set of permutations closed under taking subpermutations.
- Every pattern class $\mathcal{X}$ is defined by a minimal forbidden set B (its *basis*) which may or may not be finite.

Terminology

- Subpermutation: 3142 is a subpermutation of 5624713
- Pattern class: set of permutations closed under taking subpermutations.
- Every pattern class $\mathcal{X}$ is defined by a minimal forbidden set B (its *basis*) which may or may not be finite.
- Write $\mathcal{X} = \mathrm{Av}(B)$ (because Av stands for "avoids").

Terminology

- Subpermutation: 3142 is a subpermutation of 5624713
- Pattern class: set of permutations closed under taking subpermutations.
- Every pattern class $\mathcal{X}$ is defined by a minimal forbidden set B (its *basis*) which may or may not be finite.
- Write $\mathcal{X} = \mathrm{Av}(B)$ (because Av stands for "avoids").
- Write $\mathcal{X}_n$ for the permutations of $\mathcal{X}$ of length n.

Terminology

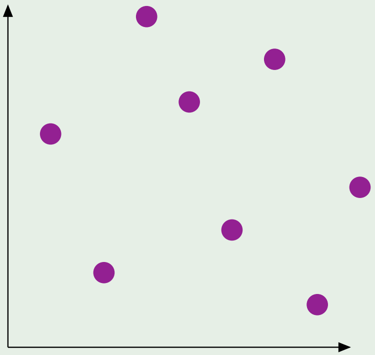
- Subpermutation: 3142 is a subpermutation of 5624713
- Pattern class: set of permutations closed under taking subpermutations.
- Every pattern class $\mathcal{X}$ is defined by a minimal forbidden set B (its *basis*) which may or may not be finite.
- Write $\mathcal{X} = \mathrm{Av}(B)$ (because Av stands for "avoids").
- Write $\mathcal{X}_n$ for the permutations of $\mathcal{X}$ of length n.
- Generating function of $\mathcal{X}$

$$f(u) = \sum_{n=0}^{\infty} |\mathcal{X}_n| u^n$$

Graphs

Example

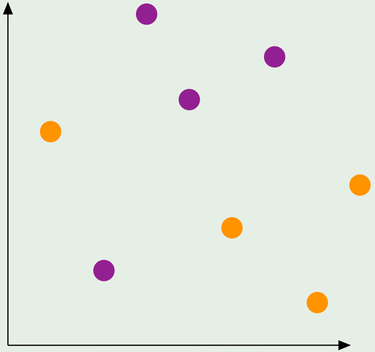
The graph of 52863714



Graphs

Example

The graph of 52863714 and subpermutation 4213



Simple permutations

- An *interval* in a permutation is a segment that contains a set of contiguous values.

Simple permutations

- An *interval* in a permutation is a segment that contains a set of contiguous values.
- Every permutation is an interval of itself, and every singleton segment is an interval.

Simple permutations

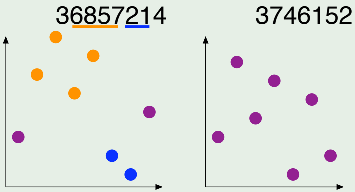
- An *interval* in a permutation is a segment that contains a set of contiguous values.
- Every permutation is an interval of itself, and every singleton segment is an interval.
- If there are no other intervals the permutation is *simple*.

Simple permutations

- An *interval* in a permutation is a segment that contains a set of contiguous values.
- Every permutation is an interval of itself, and every singleton segment is an interval.
- If there are no other intervals the permutation is *simple*.

Example

A permutation with non-trivial intervals, and a simple permutation



Substitution

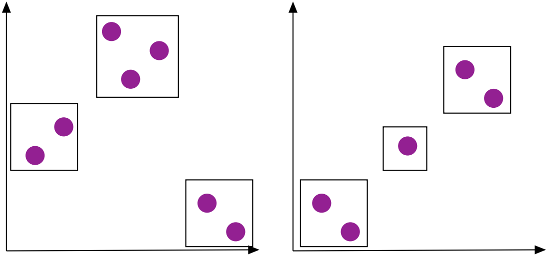
If $\tau_1, \ldots, \tau_n$ are permutations and σ is a permutation of length n then $\sigma[\tau_1, \ldots, \tau_n]$ denotes the permutation with intervals $\tau_1', \ldots, \tau_n'$ (isomorphic to $\tau_1, \ldots, \tau_n$) whose relative order is given by σ.

Substitution

If $\tau_1, \ldots, \tau_n$ are permutations and σ is a permutation of length n then $\sigma[\tau_1, \ldots, \tau_n]$ denotes the permutation with intervals $\tau'_1, \ldots, \tau'_n$ (isomorphic to $\tau_1, \ldots, \tau_n$) whose relative order is given by σ.

Example

$231[12, 312, 21] = 3475621$; $123[21, 1, 21] = 21 \oplus 1 \oplus 21 = 21354$.



The skeleton of a permutation

- Every permutation π has a representation of the form $\sigma[\tau_1, \ldots, \tau_n]$ with σ simple. The simple permutation σ is uniquely determined by π.

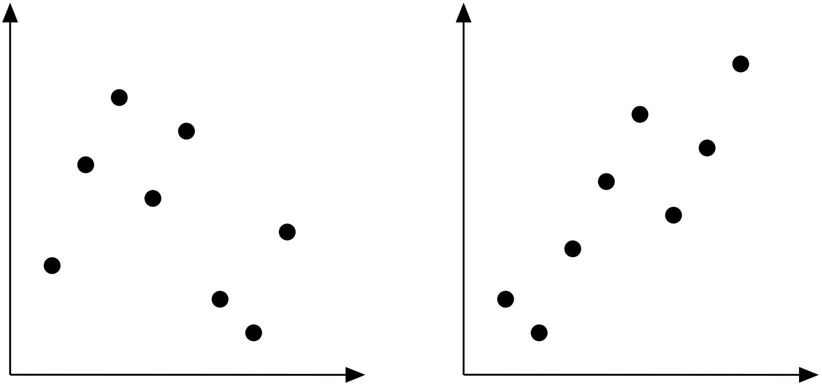
The skeleton of a permutation

- Every permutation π has a representation of the form $\sigma[\tau_1, \ldots, \tau_n]$ with σ simple. The simple permutation σ is uniquely determined by π.
- If $n > 2$, then $\tau_1, \ldots, \tau_n$ are also uniquely determined by π and then σ is the skeleton of π.

The skeleton of a permutation

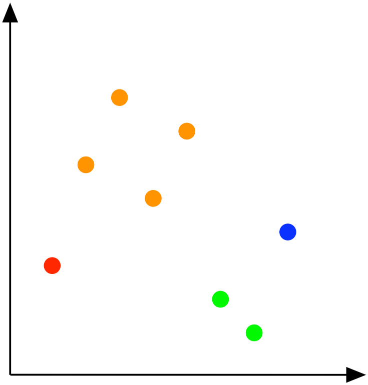
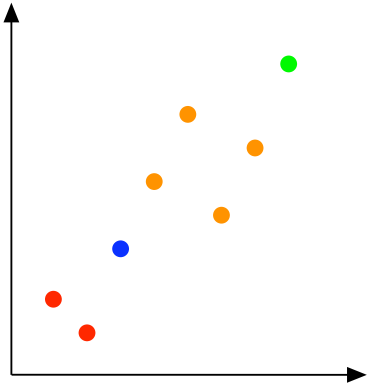
- Every permutation π has a representation of the form $\sigma[\tau_1, \ldots, \tau_n]$ with σ simple. The simple permutation σ is uniquely determined by π.
- If $n > 2$, then $\tau_1, \ldots, \tau_n$ are also uniquely determined by π and then σ is the skeleton of π.
- If $\sigma = 12$ (similarly $\sigma = 21$) write $\pi = \rho_1 \oplus \ldots \oplus \rho_k$ with k maximal, then $12 \cdots k$ is the skeleton of π

Skeleton examples



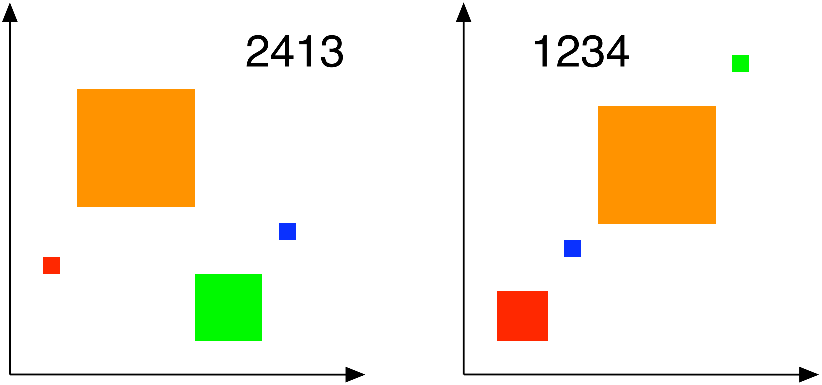
Two permutations

Skeleton examples

Two permutations and their skeletons

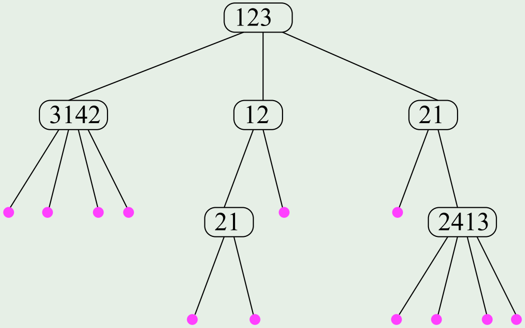
Skeleton examples



Two skeletons

Substitution decomposition

Example



The decomposition of 3142657C9B8A

Substitution closed pattern classes

- A pattern class $\mathcal{X}$ is *substitution closed* if, whenever $\sigma \in \mathcal{X}$ with $|\sigma| = n$ and $\tau_1, \ldots, \tau_n \in \mathcal{X}$, then also $\sigma[\tau_1, \ldots, \tau_n] \in \mathcal{X}$.

Substitution closed pattern classes

- A pattern class $\mathcal{X}$ is *substitution closed* if, whenever $\sigma \in \mathcal{X}$ with $|\sigma| = n$ and $\tau_1, \ldots, \tau_n \in \mathcal{X}$, then also $\sigma[\tau_1, \ldots, \tau_n] \in \mathcal{X}$.
- A pattern class is substitution closed if and only if its basis consists of simple permutations.

Substitution closed pattern classes

- A pattern class $\mathcal{X}$ is *substitution closed* if, whenever $\sigma \in \mathcal{X}$ with $|\sigma| = n$ and $\tau_1, \ldots, \tau_n \in \mathcal{X}$, then also $\sigma[\tau_1, \ldots, \tau_n] \in \mathcal{X}$.
- A pattern class is substitution closed if and only if its basis consists of simple permutations.
- An substitution closed pattern class $\mathcal{X}$ is *generated by* permutations $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ if every permutation of $\mathcal{X}$ can be obtained by iterated substitution from Γ (equivalently, $\mathcal{X}$ is the smallest substitution closed class that contains Γ).

Substitution closed pattern classes

- A pattern class $\mathcal{X}$ is *substitution closed* if, whenever $\sigma \in \mathcal{X}$ with $|\sigma| = n$ and $\tau_1, \ldots, \tau_n \in \mathcal{X}$, then also $\sigma[\tau_1, \ldots, \tau_n] \in \mathcal{X}$.
- A pattern class is substitution closed if and only if its basis consists of simple permutations.
- An substitution closed pattern class $\mathcal{X}$ is *generated by* permutations $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ if every permutation of $\mathcal{X}$ can be obtained by iterated substitution from Γ (equivalently, $\mathcal{X}$ is the smallest substitution closed class that contains Γ).
- Every substitution closed pattern class is generated by its simple permutations.

Finitely generated substitution closed classes

Theorem

Every finitely generated substitution closed pattern class is finitely based and has an algebraic generating function. Furthermore this is true for every subclass.

equivalently

Theorem

Every pattern class which has only finitely many simple permutations is finitely based and has an algebraic generating function.

Generating functions

- A pattern class with only finitely many simple permutations and that avoids some $k, k-1, \ldots, 1$ has a rational generating function.

Generating functions

- A pattern class with only finitely many simple permutations and that avoids some $k, k-1, \ldots, 1$ has a rational generating function.
- A pattern class whose permutations contain at most d copies of 231 (for some d) has an algebraic generating function [Bóna, 1997].

Generating functions

- A pattern class with only finitely many simple permutations and that avoids some $k, k-1, \ldots, 1$ has a rational generating function.
- A pattern class whose permutations contain at most d copies of 231 (for some d) has an algebraic generating function [Bóna, 1997].
- Every proper subclass of $\mathrm{Av}(231)$ has a rational generating function.

Growth rates

The *growth rate* of a class with generating function $f(x) = \sum_{n=0}^{\infty} v_n x^n$ is the limit (if it exists)

$$\lim_{n\to\infty} \sqrt[n]{v_n}$$

Growth rates

The *growth rate* of a class with generating function $f(x) = \sum_{n=0}^{\infty} v_n x^n$ is the limit (if it exists)

$$\lim_{n\to\infty} \sqrt[n]{v_n}$$

Conjecture

Every pattern class has a growth rate.

Growth rates

Put $\iota_k = 12 \cdots k$ and $\delta_k = k \cdots 21$.

- The growth rate of a class $\mathrm{Av}(\delta_k, \iota_p \oplus 231 \oplus \iota_q)$ is independent of p and q

Growth rates

Put $\iota_k = 12 \cdots k$ and $\delta_k = k \cdots 21$.

- The growth rate of a class $\mathrm{Av}(\delta_k, \iota_p \oplus 231 \oplus \iota_q)$ is independent of p and q
- The growth rate of a class $\mathrm{Av}(\delta_k, \iota_p \oplus 2413 \oplus \iota_q, \iota_r \oplus 3142 \oplus \iota_s)$ is independent of p, q, r, s

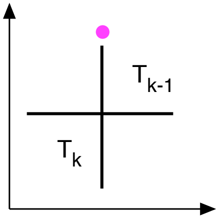
Growth rates

Put $\iota_k = 12\cdots k$ and $\delta_k = k\cdots 21$.

- The growth rate of a class $\mathrm{Av}(\delta_k, \iota_p \oplus 231 \oplus \iota_q)$ is independent of p and q
- The growth rate of a class $\mathrm{Av}(\delta_k, \iota_p \oplus 2413 \oplus \iota_q, \iota_r \oplus 3142 \oplus \iota_s)$ is independent of p, q, r, s
- The proofs of both these results begin with proving that these pattern classes have only finitely many simple permutations

Finding the growth rate of $T_k = \mathrm{Av}(\delta_k, 231)$

Let $t_k(x)$ be the generating function of T_k



Non-empty permutation of T_k

Hence $t_k = 1 + xt_k t_{k-1}$ which gives

$$t_k = \frac{1}{1 - xt_{k-1}}$$

Finding the growth rate of $T_k = \mathrm{Av}(\delta_k, 231)$

Then t_k is a rational function q_{k-1}/q_k of x where $q_1 = 1, q_2 = 1 - x$ and, for $k > 2$,

$$q_k = q_{k-1} - xq_{k-2}$$

Finding the growth rate of $T_k = \mathrm{Av}(\delta_k, 231)$

Then t_k is a rational function q_{k-1}/q_k of x where $q_1 = 1, q_2 = 1 - x$ and, for $k > 2$,

$$q_k = q_{k-1} - xq_{k-2}$$

Then

$$\begin{aligned} q_k &= \frac{(1+\sqrt{1-4x})^{k+1} - (1-\sqrt{1-4x})^{k+1}}{2^{k+1}\sqrt{1-4x}} \\ &= \sum_i \binom{k-i}{i}(-x)^i \end{aligned}$$

Finding the growth rate of $T_k = \mathrm{Av}(\delta_k, 231)$

Theorem

The growth rate of the classes $\mathrm{Av}(\delta_k, \iota_p \oplus 231 \oplus \iota_q)$ *is*

$$2 + 2\cos\left(\frac{2\pi}{k+1}\right)$$

Proof.

Solve $q_k(x) = 0$ for smallest root (requires taking a $(k+1)^{\mathrm{th}}$ root) and use reciprocal. □

Finding the growth rate of $U_k = \mathrm{Av}(\delta_k, 2413, 3142)$

The same technology to find the generating function is much messier. For example U_4 has generating function

$$\frac{x\left(1-5x+11x^2-11x^3+7x^4-3x^5+x^6\right)}{1-7x+19x^2-28x^3+23x^4-12x^5+4x^6-x^7}$$

Finding the growth rate of $U_k = \mathrm{Av}(\delta_k, 2413, 3142)$

The same technology to find the generating function is much messier. For example U_4 has generating function

$$\frac{x\left(1-5x+11x^2-11x^3+7x^4-3x^5+x^6\right)}{1-7x+19x^2-28x^3+23x^4-12x^5+4x^6-x^7}$$

The degree Δ_k of the denominator of U_k rises rapidly with k. It appears that this denominator is always irreducible and that

$$\Delta_k = 1 + \sum_{i=2}^{k-1} \left\lfloor \frac{k-1}{i-1} \right\rfloor \Delta_i$$

k	2	3	4	5	6	7	8	9	10	11	12
Δ_k	1	3	7	18	37	85	171	364	736	1513	3027

Density and growth rate of simple permutations

- On average one permutation in every 7.389... ($= e^2$) is simple.

Density and growth rate of simple permutations

- On average one permutation in every 7.389... $(= e^2)$ is simple.
- What is the density of simple permutations in a particular pattern class?

Density and growth rate of simple permutations

- On average one permutation in every 7.389... ($= e^2$) is simple.
- What is the density of simple permutations in a particular pattern class?
- Does the set of simple permutations in a pattern class always have a well-defined growth rate?

Density and growth rate of simple permutations

- On average one permutation in every 7.389... $(= e^2)$ is simple.
- What is the density of simple permutations in a particular pattern class?
- Does the set of simple permutations in a pattern class always have a well-defined growth rate?
- How does that compare with the growth rate of the pattern class?

Examples - Lutful Karim

- The number of simple permutations in $\mathrm{Av}(321, 4123)$ satisfies

$$x_n = x_{n-2} + x_{n-3}$$

Examples - Lutful Karim

- The number of simple permutations in $\mathrm{Av}(321, 4123)$ satisfies

$$x_n = x_{n-2} + x_{n-3}$$

- Known answers for every $\mathrm{Av}(\alpha, \beta)$ with $|\alpha| = 3, |\beta| = 4$

Examples - Lutful Karim

- The number of simple permutations in $\mathrm{Av}(321, 4123)$ satisfies

$$x_n = x_{n-2} + x_{n-3}$$

- Known answers for every $\mathrm{Av}(\alpha, \beta)$ with $|\alpha| = 3, |\beta| = 4$
- The number of simple permutations of length n in $\mathrm{Av}(4321, 2413)$ is polynomial in n (two slightly different cubics for the even and odd cases)

Examples - Lutful Karim

- The number of simple permutations in $\mathrm{Av}(321, 4123)$ satisfies

$$x_n = x_{n-2} + x_{n-3}$$

- Known answers for every $\mathrm{Av}(\alpha, \beta)$ with $|\alpha| = 3, |\beta| = 4$
- The number of simple permutations of length n in $\mathrm{Av}(4321, 2413)$ is polynomial in n (two slightly different cubics for the even and odd cases)

For which pattern classes does the number of simple permutations of length n grow as a polynomial? as a constant?

Principal pattern classes

Definition

A pattern class of the form $\mathrm{Av}(\alpha)$ is called a *principal* class.

Consider the substitution closure of such a class.

- For which α is it finitely based?
- For which α is it finitely generated?

Principal pattern classes

Definition

A pattern class of the form $\mathrm{Av}(\alpha)$ is called a *principal* class.

Consider the substitution closure of such a class.

- For which α is it finitely based?
- For which α is it finitely generated?

Theorem

The substitution closure of $\mathrm{Av}(\alpha)$ *is finitely generated if and only if* $\alpha \in \{1, 12, 21, 132, 213, 231, 312\}$

The finite basis question is much more subtle!

Finding the basis of the substitution closure of $\mathrm{Av}(\alpha)$

- A simple permutation belongs to $\mathrm{Av}(\alpha)$ if and only if it belongs to the substitution closure

Finding the basis of the substitution closure of $\mathrm{Av}(\alpha)$

- A simple permutation belongs to $\mathrm{Av}(\alpha)$ if and only if it belongs to the substitution closure
- The simple permutations not in the substitution closure are precisely the simple permutations that contain α

Finding the basis of the substitution closure of $\mathrm{Av}(\alpha)$

- A simple permutation belongs to $\mathrm{Av}(\alpha)$ if and only if it belongs to the substitution closure
- The simple permutations not in the substitution closure are precisely the simple permutations that contain α
- Hence the basis permutations of the substitution closure are precisely the minimal simple extensions of α

Finite and infinite types

Definition

α has *finite type* if the substitution closure of $\mathrm{Av}(\alpha)$ is finitely based; otherwise α has *infinite type*.

Finite and infinite types

Definition

α has *finite type* if the substitution closure of $\mathrm{Av}(\alpha)$ is finitely based; otherwise α has *infinite type*.

Question

How do we distinguish the permutations of finite type from those of infinite type?

Finite types with skeleton 12

Up to symmetry the only finite types of the form $\alpha \oplus \beta$ with α, β indecomposable are

- $\alpha, \beta \in \{1, 21, 312, 231\}$
- $2413 \oplus 1 = 24135$

Finite types with skeleton 123

Up to symmetry the only finite types of the form $\alpha \oplus \beta \oplus \gamma$ are

- $1 \oplus 1 \oplus 1 = 123$
- $1 \oplus 1 \oplus 21 = 1243$
- $1 \oplus 21 \oplus 1 = 1324$
- $21 \oplus 1 \oplus 21 = 21354$

Finite types with skeleton 1234. . .

Up to symmetry the only finite types of the form $\alpha \oplus \beta \oplus \gamma \oplus \delta \oplus \ldots$ are

- None

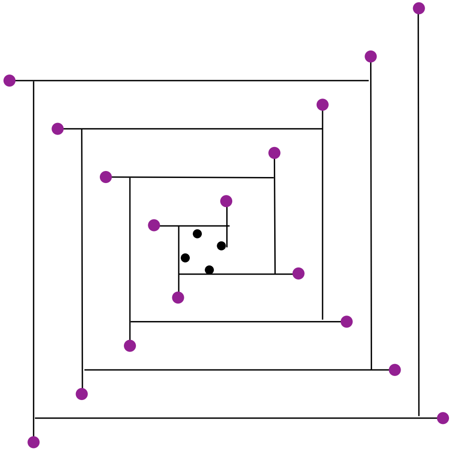
Finite types with simple skeleton of length at least 4

If the skeleton of π is a simple permutation of length more than 2 and the associated intervals are all among

$$\{1, 12, 21, 132, 213, 231, 312\}$$

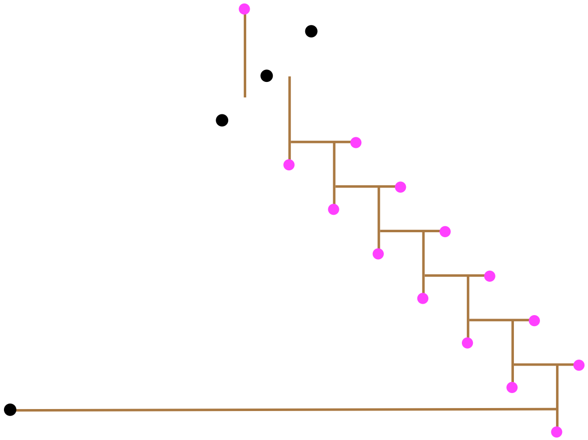
then π also has finite type.

The other finite types discounting symmetries



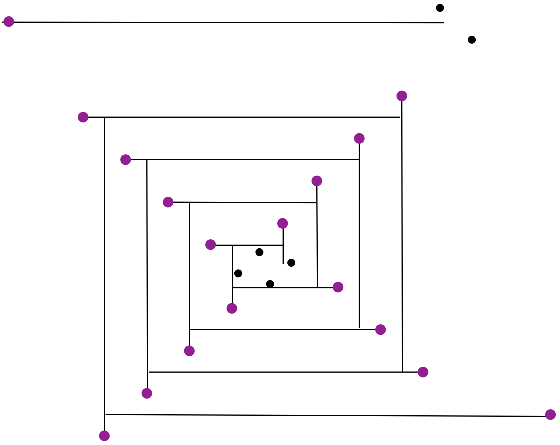
A permutation of finite type

1234 has infinite type



Minimal simple extensions of 1234

241365 has infinite type



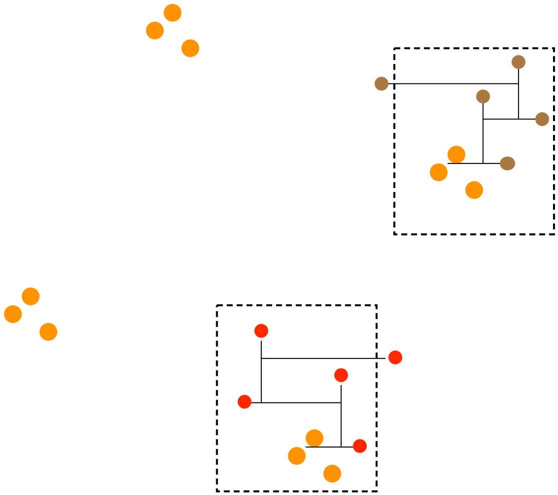
Minimal simple extensions of 241365

Proving that a permutation has finite type

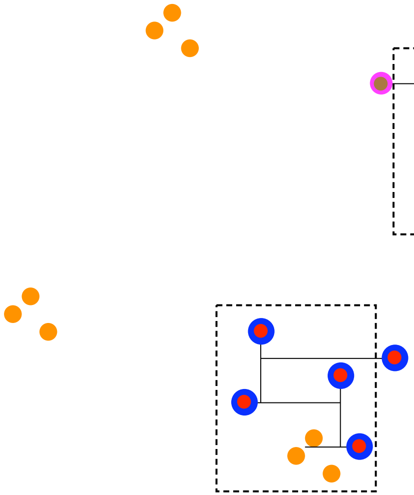
- Potential intervals in simple permutations define pin sequences
- The simple extensions of a (non-simple) permutation all have pin sequences that intersect in various ways. If these pin sequences are sufficiently long...

Proving that a permutation has finite type

Proving that a permutation has finite type



Proving that a permutation has finite type



Some references

Michael Albert, Mike Atkinson
Simple permutations and pattern restricted permutations,
Discrete Mathematics, 300 (2005), 1-15.

Miklos Bóna
The number of permutations with exactly r 132-subsequences is P-recursive in the size!,
Advances in Applied Mathematics, 18 (1997), 510-522.

Robert Brignall, Sophie Huczynska, Vince Vatter
Simple permutations and algebraic generating functions,

Toufik Mansour, Alex Vainshtein
Restricted permutations and Chebyshev polynomials,
Sem. Lothar. Combin. (2001) 47, B47c.