$(3 + 1)$-avoiding permutations

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Our Result

- We have explicitly described the form of permutations in the pattern class $\mathrm{Av}(4123, 2341)$
- and enumerated it

Outline of talk

1. Introduction
   - Motivation
   - Proof strategy
2. Describing the simple permutations
   - The role of 3412
   - Simple permutations in $\mathrm{Av}(2341, 4123, 3412)$
3. The enumeration formulae

Two reasons for analysing $\mathrm{Av}(2341, 4123)$

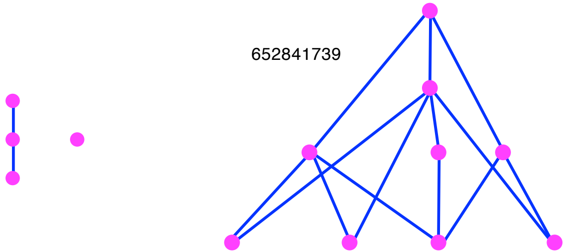
Reason 1

- The first moderately difficult pattern classes to be enumerated were of the form $\mathrm{Av}(\alpha, \beta)$ with $|\alpha| = |\beta| = 4$.
- There are 56 inequivalent pattern classes of the form $\mathrm{Av}(\alpha, \beta)$ with $|\alpha| = |\beta| = 4$.
- These 56 classes fall into 38 Wilf classes
- Just over half of these have been enumerated
- These pattern classes are "on the cusp" of what our present techniques can achieve

Two reasons for analysing $\mathrm{Av}(2341, 4123)$

Reason 2

$\mathrm{Av}(2341, 4123)$ is associated with posets that are (3+1)-free



The poset 3+1 and a poset not containing it

Proof strategy

- Find the simple permutations in $\mathrm{Av}(2341, 4123)$
- Find all possible inflations of the simple permutations

Definition

A simple permutation is one without any non-trivial intervals

Example

38157462 has an interval but 35142 is simple.

Theorem

Every permutation arises from a unique simple permutation by inflating points into intervals.

Inflations of simple permutations in $\mathrm{Av}(2341, 4123)$

Theorem

Every permutation of $\mathrm{Av}(2341, 4123)$*, except for sums and skew sums, is an inflation of a simple permutation whose points have been inflated by decreasing sequences.*

Two types of simple permutation

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A simple permutation of $\mathrm{Av}(2341, 4123)$ *that contains 123 either avoids 3412 or is* 5274163.

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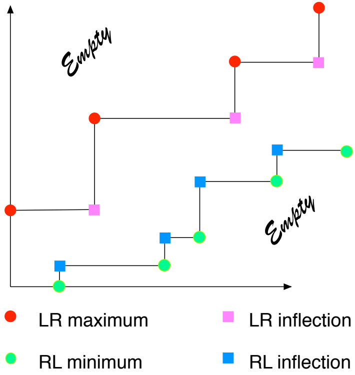
So we can analyse the simple permutations as essentially those that avoid 123 or those that avoid 3412. Since $\mathrm{Av}(123)$ is a (moderately) well understood pattern class this means that we have to concentrate on simple permutations in $\mathrm{Av}(2341, 4123, 3412)$.

LR maxima and RL minima

Definition

If π is a permutation then $\pi(i)$ is a LR maximum if $\pi(i)$ is larger than all of $\pi(1), \ldots, \pi(i-1)$. Similarly $\pi(i)$ is a RL minimum if $\pi(i)$ is smaller than any of $\pi(i+1), \pi(i+2), \ldots$



Inflection properties

For a simple permutation in $\mathrm{Av}(2341, 4123, 3412)$

1. The set of inflections is an increasing set
2. The LR inflections alternate with the RL inflections

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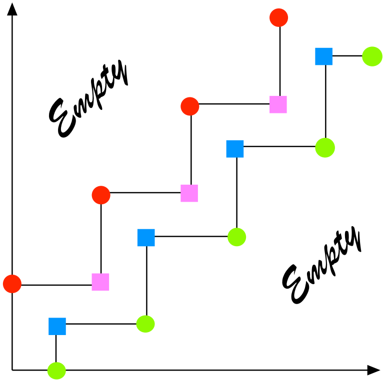
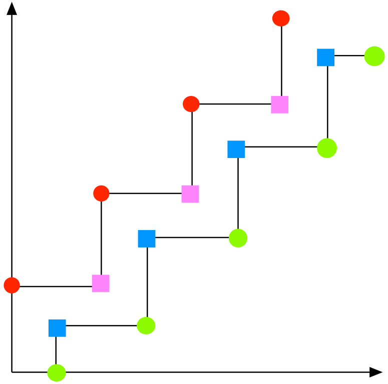


Cell decomposition



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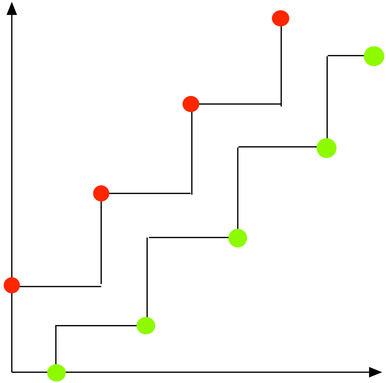

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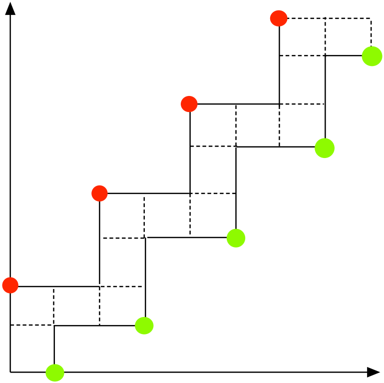

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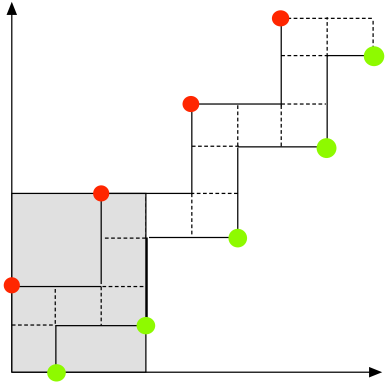

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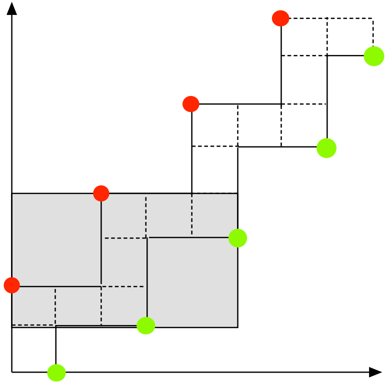

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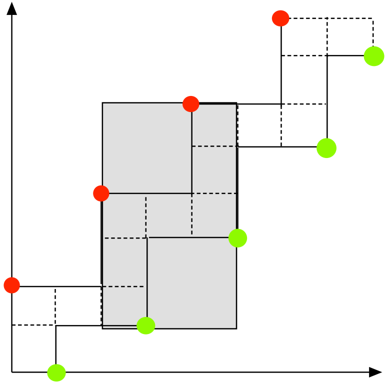

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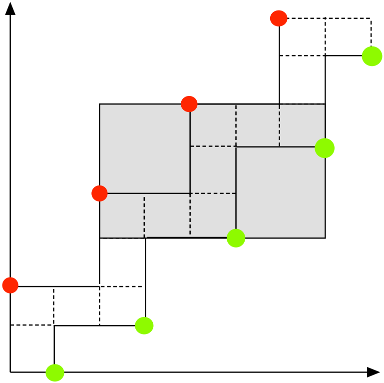

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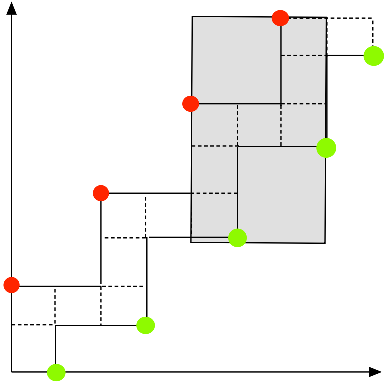

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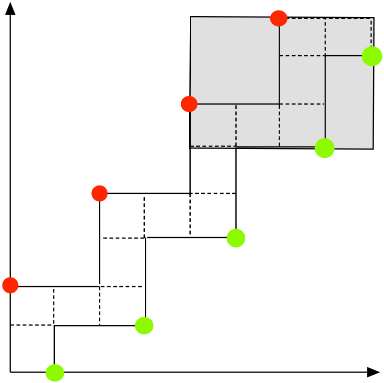
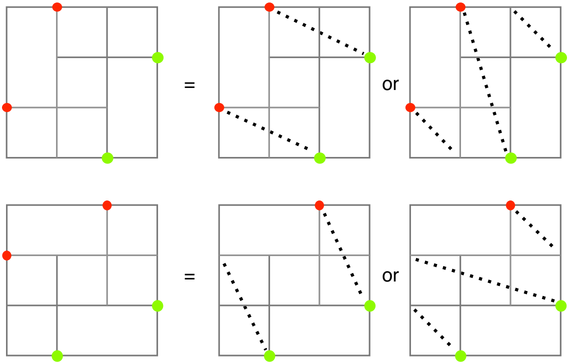


Cell types



The bottom line

Theorem

The generating function f *for* $\mathrm{Av}(2341, 4123)$ *has the form* $f = 1/(1-g)$ *where*

$$g = \frac{(1 - 2x - \sqrt{1-4x})}{2x} - \frac{r}{s}$$

and

$$\begin{aligned} r &= (1 - 13x + 74x^2 - 247x^3 + 539x^4 - 805x^5 + 834x^6 - \\ &\quad 595x^7 + 283x^8 - 80x^9 + 8x^{10})x^2 \\ s &= (1-x)^7(1-2x)(1 - 6x + 12x^2 - 9x^3 + x^4) \end{aligned}$$

Questions?