Permutation Patterns

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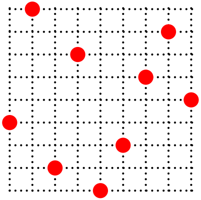
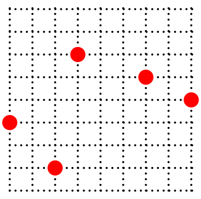
Permutation classes

Definition

A permutation class is a collection of permutations, $\mathcal{C}$, with the property that, if $\pi \in \mathcal{C}$ and we erase some points from its plot, then the permutation defined by the remaining points is also in $\mathcal{C}$.



$492713685 \in \mathcal{C}$ implies



$21543 \in \mathcal{C}$

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- If X is a set of permutations, then $\mathrm{Av}(X)$ is the permutation class consisting of those permutations which do not dominate any permutation of X
- We can also describe classes positively: e.g. the class of all permutations which consist of two or fewer increasing runs

Origin stories

- Modern interest in permutation patterns can be traced back to work of Knuth, Pratt and Tarjan on “partial sorting operators”
- For example, consider processing an incoming stream of data items labelled 1 through *n* (in arbitrary order) using a single stack, and trying to output them in sorted order

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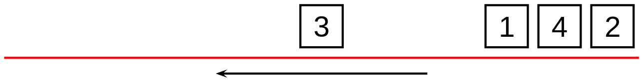

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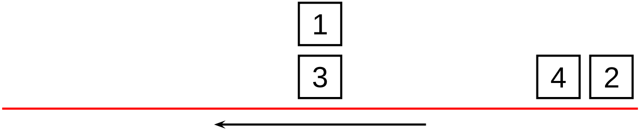

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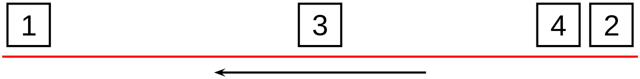

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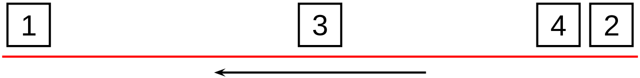

Three questions

- How many permutations of length n can be sorted by a stack?
- What patterns do they avoid?
- What do they look like?

Three answers

- A permutation π can be sorted by a stack if and only if $\pi = \alpha n \beta$ where every element of α is smaller than every element of β and α and β can be sorted by a stack
- These are exactly the permutations in Av(231)
- The generating function, f, for this class of permutations satisfies:

$$\begin{aligned} f &= (1+f)x(1+f) \\ &= x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + \cdots \\ &= \frac{1 - 2x - \sqrt{1-4x}}{2x} \end{aligned}$$

Three more questions

Permutations that can be written as a concatenation of two increasing runs are exactly those sortable by “split the permutation in two somewhere then merge the two pieces”

- What are the minimal permutations not expressible as a concatenation of two increasing runs?
- How many permutations of that type are there?
- What can be said about the enumeration of proper subclasses of this class?

Stanley-Wilf conjecture

Relative to the set of all permutations, proper permutation classes are small. Specifically:

Theorem

Let $\mathcal{C}$ be a proper permutation class. Then, the growth rate of $\mathcal{C}$,

$$\mathrm{gr}(\mathcal{C}) = \limsup |\mathcal{C} \cap \mathcal{S}_n|^{1/n}$$

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But still, interesting permutation classes all have growth rate larger than 2. Investigating such classes exhaustively is impossible, and there are no known general methods for sampling in a uniform, or near-uniform manner.

Simple permutations

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- Simple permutations form a positive proportion of all permutations (asymptotically $1/e^2$)
- In many (conjecturally all) proper permutation classes they have density 0
- We can hope to understand a class by understanding its simples and how they *inflate*
- Specifically, this may yield functional equations of the generating function and hence computations of the enumeration and/or growth rate

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Finitely many simple permutations

Theorem (A and Atkinson, 2005)

If a class has only finitely many simple permutations then it has an algebraic generating function.

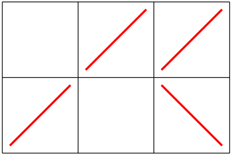
The method is effective, “en principe”, so we’re done with those classes!

Av(4231, 1324)

- Considered in A, At and Vatter “Counting 1324, 4231-Avoiding Permutations”, EJC 16, R136
- Main ideas: identify some features in the simple permutations
- Then worry about the details

$\mathrm{Av}(4312, 3142)$

- Every simple permutation in this class looks like:



- This yields a regular language for the simple permutations
- The allowed inflations of these permutations are easily described, yielding a recursive description of the class
- This leads to an equation for its generating function:

$$\begin{aligned}(x^3 - 2x^2 + x)f^4 &+ (4x^3 - 9x^2 + 6x - 1)f^3 \\ &+ (6x^3 - 12x^2 + 7x - 1)f^2 \\ &+ (4x^3 - 5x^2 + x)f \\ &+ x^3 \quad = 0\end{aligned}$$