Permutation Patterns

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Outline of talk

1. Permutations and Patterns: basic concepts
   - Permutations and graphs
   - Origins
2. Pattern classes
   - Enumeration
   - Structure

Permutations

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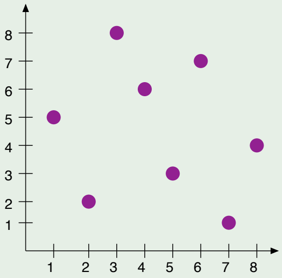
Example

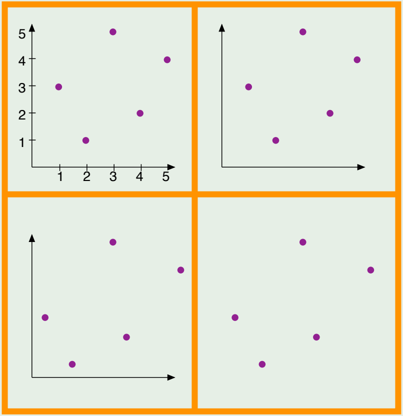
The graph of 52863714



The cardinal sin: unlabeled axes

Example

Increasingly sloppy graphs of 31524



Subpermutations

Example

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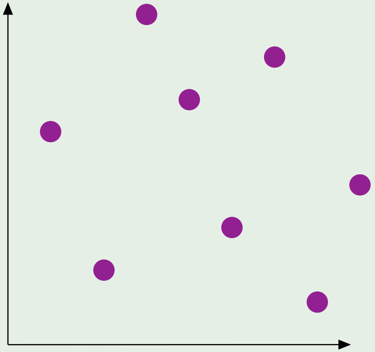
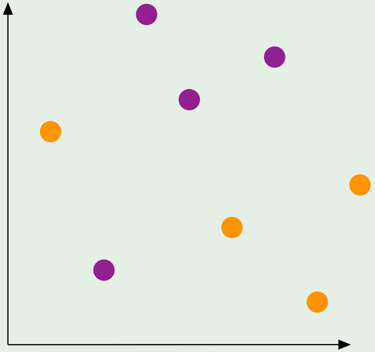


Subpermutations

Example

The graph of 52863714 and subpermutation 4213



Formalities

Definition

π is a subpermutation of a permutation $\sigma = s_1 \cdots s_n$ if σ has a subsequence whose terms are ordered relatively the same as π.

Example

4213 is a subpermutation of 52863714

Notation

$4213 \preceq 52863714$ *and* $3214 \not\preceq 52863714$

The $\preceq$ relation is a partial order on the set of all permutations

Origins: Erdös - Szekeres Theorem

Theorem

If σ is a sequence of distinct real numbers that has no increasing subsequence of length r nor decreasing subsequence of length s then its length is no more than $(r-1)(s-1)$.

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In the language of permutations:

Theorem

If σ is a permutation and neither $12\cdots r$ nor $s\cdots 21$ is a subpermutation of σ then $|\sigma| \leq (r-1)(s-1)$.

Origins: Stack sorting

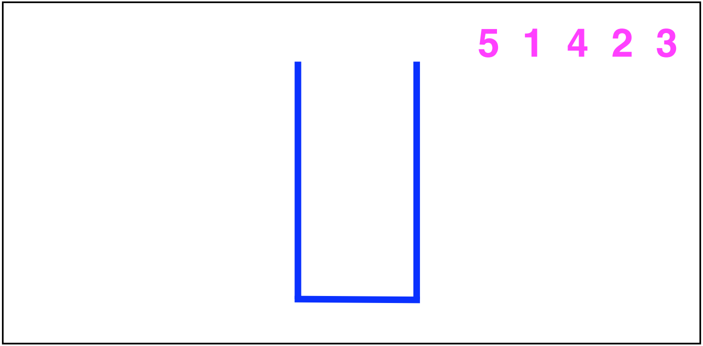
Definition

A *Stack* is a list, one end of which is called the top of the list. Items can be inserted at the top and removed from the top of the list.

Question

Which permutations can be sorted via a stack?

Sorting a permutation



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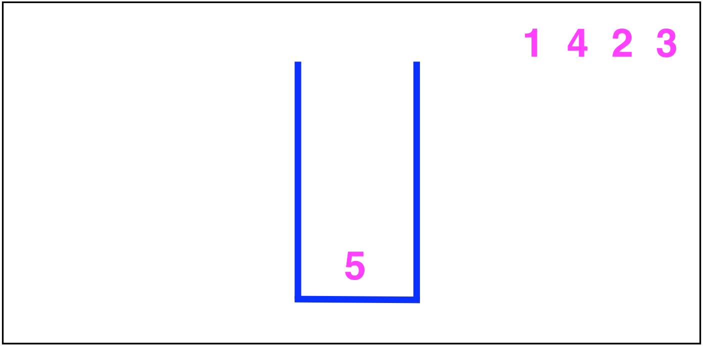

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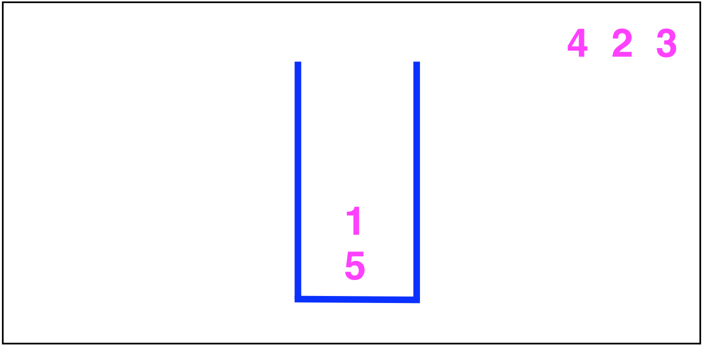

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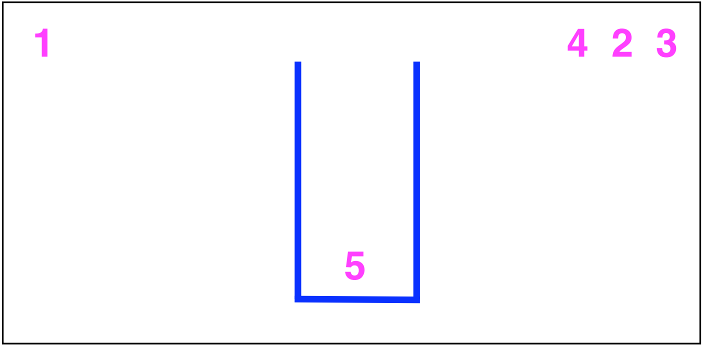

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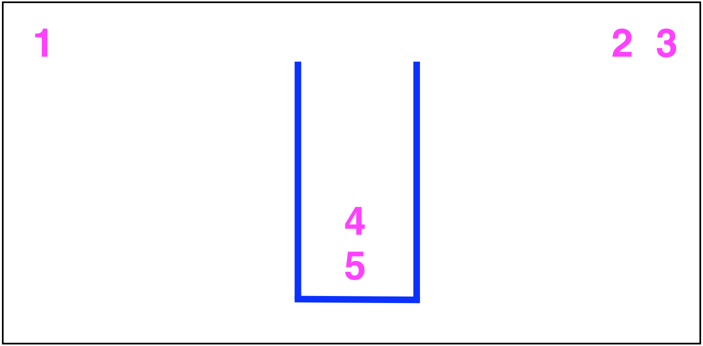

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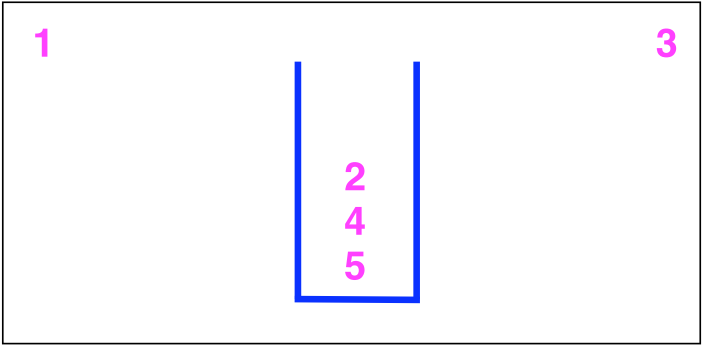

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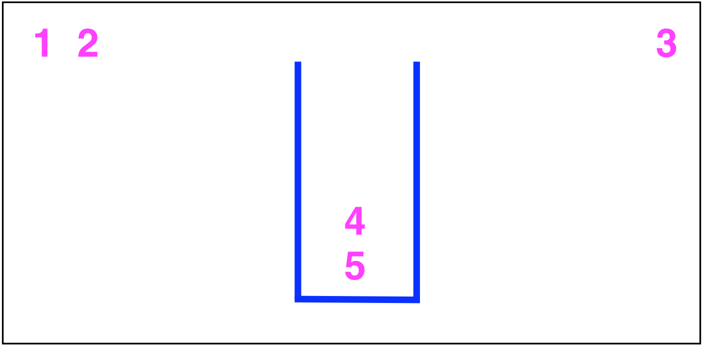

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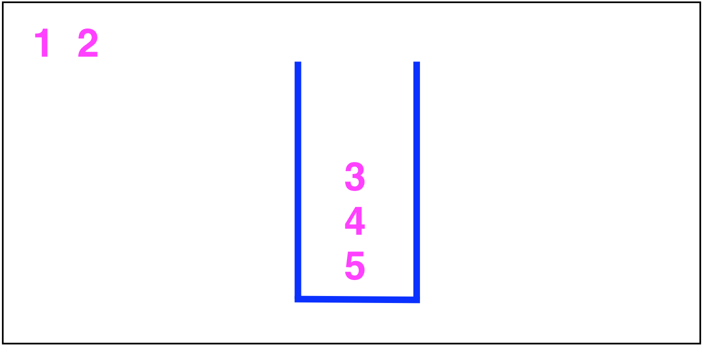

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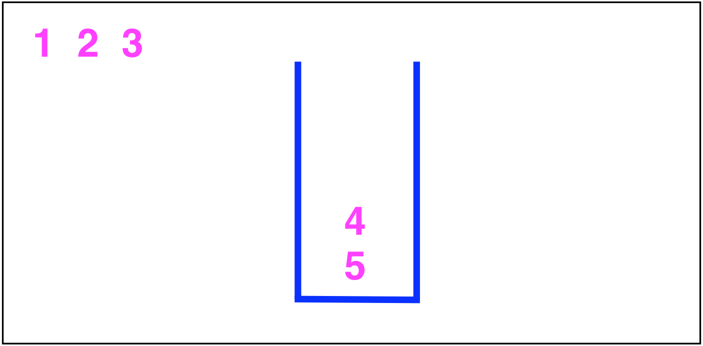

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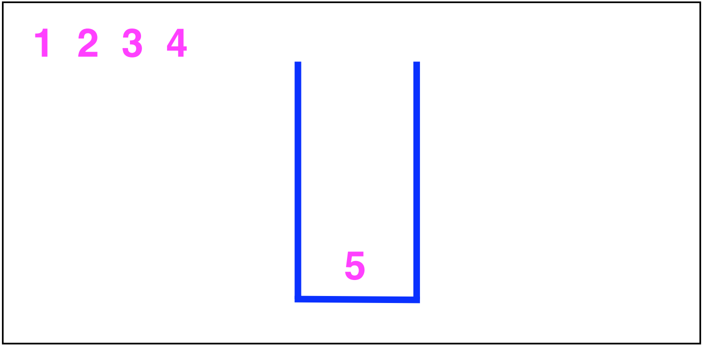

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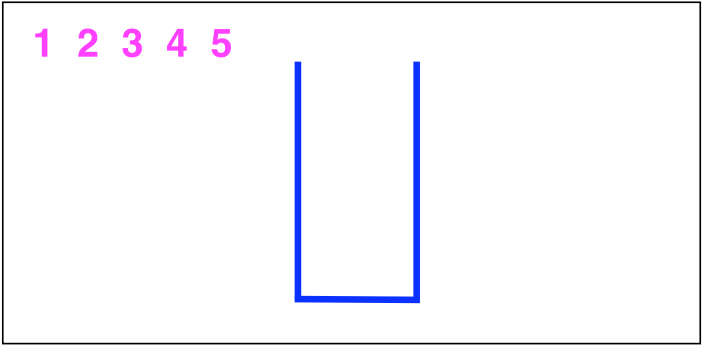

Knuth theorems

Theorem

A permutation σ can be sorted via a stack if and only if $231 \not\preceq \sigma$.

Theorem

There are

$$\frac{\binom{2n}{n}}{n+1}$$

stack sortable permutations of length n.

Pattern classes

Definition

A *pattern class* is a set of permutations closed under taking subpermutations (down-set in the partial order)

Every pattern class $\mathcal{X}$ can be defined by a set of avoided permutations.

Notation

$\mathcal{X} = \mathrm{Av}(B)$ *means that* $\mathcal{X}$ *is the pattern class defined by the avoiding the permutations in the set* B.

Theorem

1. $\mathrm{Av}(12 \cdots r, s \cdots 21)$ *is finite*
2. $\mathrm{Av}(231)$ *is the set of stack-sortable permutations.*

Enumeration

A huge section of pattern class research asks:

Given a set B of permutations, how many permutations of length n does $\mathrm{Av}(B)$ have?

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Answers to define the enumeration sequence (s_n) are of the forms

- a formula for s_n
- the generating function $\sum s_n x^n$
- asymptotic upper and lower bounds on s_n

The more permutations B has, and the shorter they are, the more likely it is that we can solve the enumeration problem.

Examples of enumeration results

Knuth For $\mathrm{Av}(\beta)$, with $|\beta| = 3$, $s_n = \binom{2n}{n}/(n+1)$

Simion et al. For $\mathrm{Av}(\alpha, \beta)$ with $|\alpha| = 3$ and $|\beta| = 3$ formulae for s_n are known in all cases

Boná For $\mathrm{Av}(1342)$

$$\sum s_n x^n = \frac{32x}{1 + 20x - 8x^2 - (1-8x)^{3/2}}$$

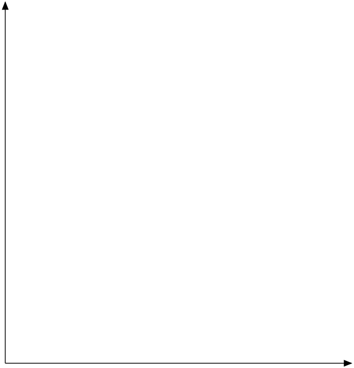
Albert et al. For $\mathrm{Av}(1324)$, $s_n = \Omega(9.47^n)$

MDA, West For $\mathrm{Av}(\alpha, \beta)$ with $|\alpha| = 3$ and $|\beta| = 4$ formulae for s_n are known in all cases

Enumeration frontier

- No exact enumerations of $\mathrm{Av}(\beta)$ for any $|\beta| \geq 5$
- About half of the cases $\mathrm{Av}(\alpha, \beta)$ with $|\alpha| = |\beta| = 4$ have been enumerated
- Sporadic results only for $\mathrm{Av}(\alpha, \beta, \gamma, \ldots)$

An example proof. Enumerating $\mathrm{Av}(231, 312)$: diagram-chasing



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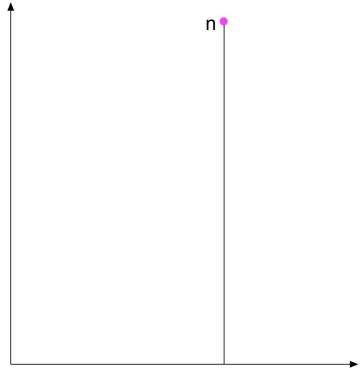

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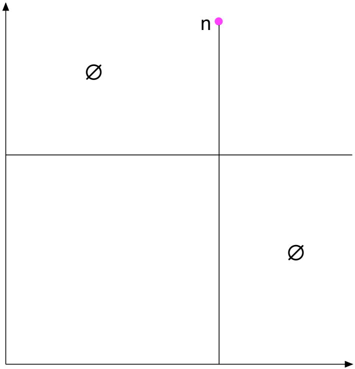

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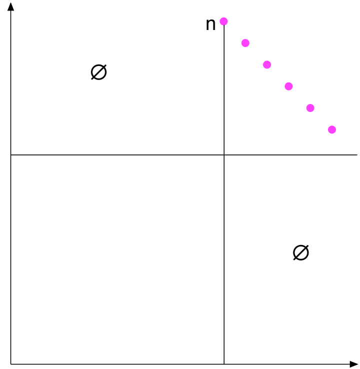

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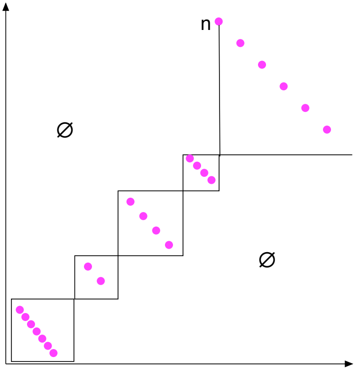

So permutations of $\mathrm{Av}(231, 312)$ are determined by compositions of n; hence 2^{n-1} of them.

The Marcus-Tardos Theorem (2004)

Theorem

Let s_n *be the number of permutations of length* n *in a proper pattern class* $\mathcal{X}$*. Then there is a constant* $\kappa = \kappa(\mathcal{X})$ *such that*

$$s_n \leq \kappa^n$$

Conjecture

Let s_n *be the number of permutations of length* n *in a proper pattern class* $\mathcal{X}$*. Then*

$$\lim_{n\to\infty} \sqrt[n]{s_n}$$

exists.

A structure theory for pattern classes?

We recognise structure when we see it: e.g.

- The intersection and union of pattern classes is again a pattern class – many ways of making new pattern classes out of old.
- The subpermutation order is not a quasi-well-order; but some pattern classes are quasi-well-ordered (e.g. $\mathrm{Av}(231)$ is quasi-well-ordered whereas $\mathrm{Av}(321)$ is not)
- Often we can describe a pattern class other than by saying "it has the following restrictions..."

Simple permutations

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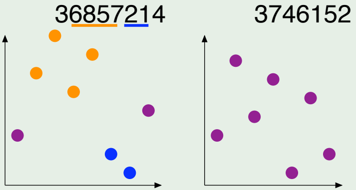
Example

A permutation with non-trivial intervals, and a simple permutation



Inflation

If σ is a permutation of length n and $\tau_1, \ldots, \tau_n$ are permutations then the *inflation* of σ by $\tau_1, \ldots, \tau_n$ (denoted by $\sigma[\tau_1, \ldots, \tau_n]$) is the permutation with intervals $\tau'_1, \ldots, \tau'_n$ (isomorphic to $\tau_1, \ldots, \tau_n$) whose relative order is given by σ.

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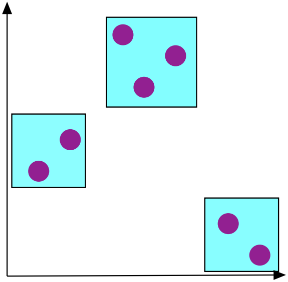
Example

$231[12, 312, 21] = 3475621.$



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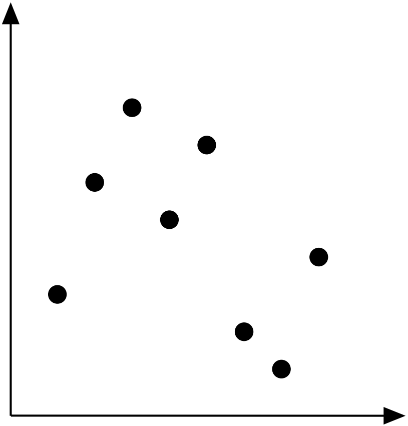

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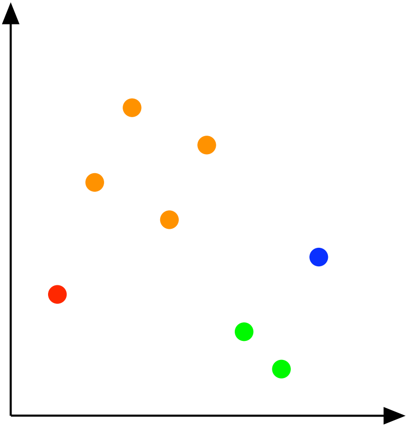

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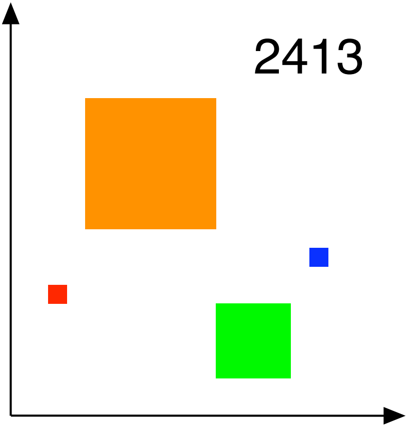


A two step process

To understand a pattern class $\mathcal{X}$

- Find all its simple permutations
- For each simple permutation find all inflations that lie in $\mathcal{X}$

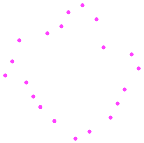
The easiest case of this approach is when the number of simple permutations is finite: in that case (MDA, Albert)

- $\mathcal{X}$ and every subclass has an algebraic generating function
- $\mathcal{X}$ is quasi-well-ordered
- Every subclass of $\mathcal{X}$ is defined by a finite number of restrictions

A more challenging example: $\mathrm{Av}(1324, 4231)$

How does one set about enumeration?

1. Find the simple permutations
   1. The simple permutations 25314 and 41352 can only be subpermutations in a very particular way
   2. Every other simple permutation has the rough diamond form



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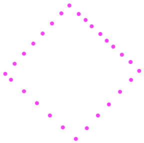

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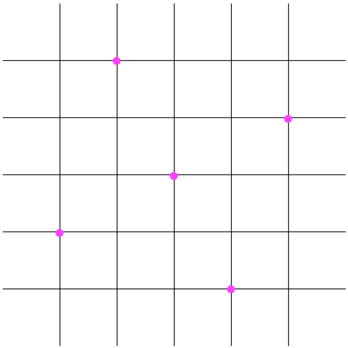
Diagram Chase: permutations in $\mathrm{Av}(1324, 4231)$ containing 25314



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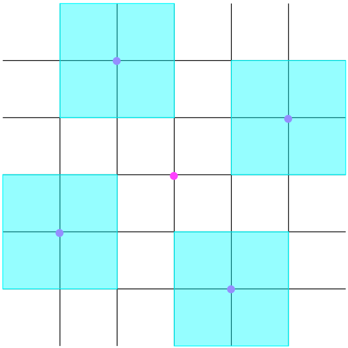

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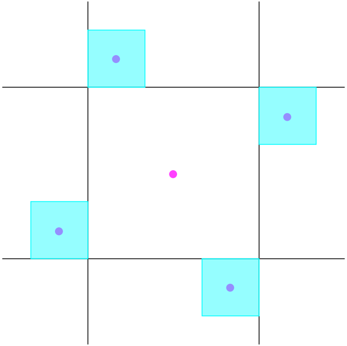

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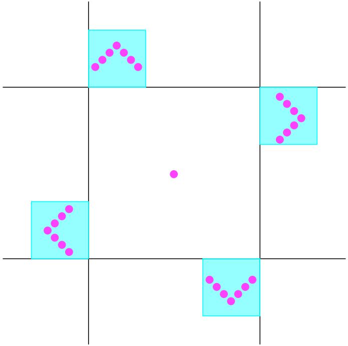
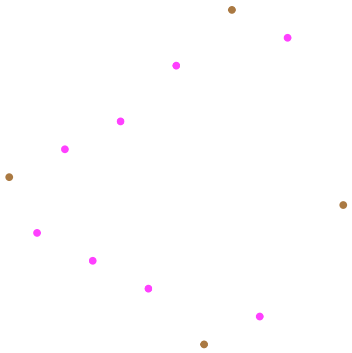


Inflations of diamonds



The simple permutation 7 5 8 4 9 3 10 1 12 2 11 6

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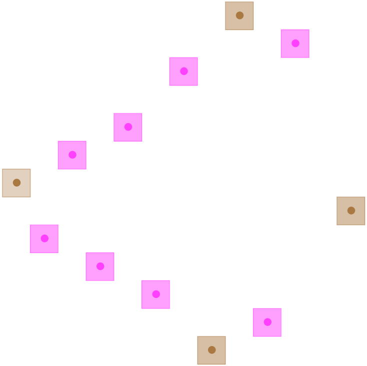

The simple permutation 7 5 8 4 9 3 10 1 12 2 11 6 - inflated - how?

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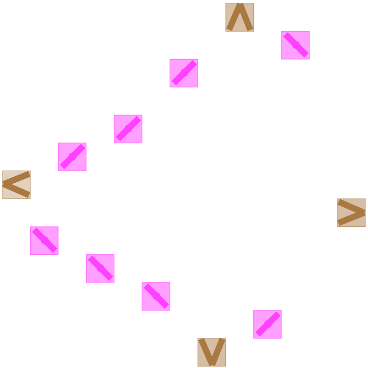

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The bottom line - MDA, Albert, Vatter

Theorem

The generating function for $\mathrm{Av}(1324, 4231)$ *is*

$$\frac{1 - 12x + 59x^2 - 152x^3 + 218x^4 - 168x^5 + 58x^6 - 6x^7}{(1-x)(1-2x)^4(1-4x+2x^2)}.$$