Young Classes of Permutations

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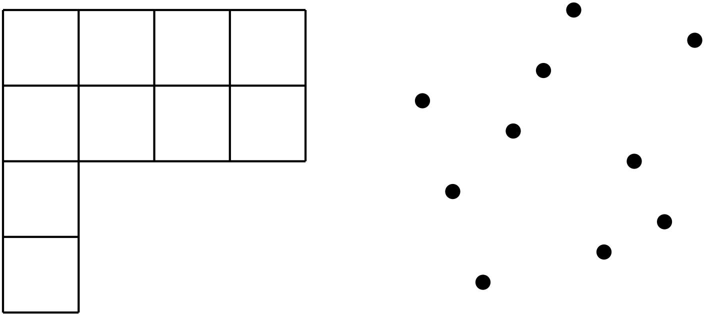
http://arxiv.org/abs/1008.4615

The shape of a permutation

7 4 1 6 8 10 2 5 3 9



The big question

- Two permutations are *shape equivalent* if their Young diagrams have the same shape.
- A permutation σ is *involved* in a permutation π if π has a subsequence order isomorphic to σ.
- A *permutation class* is a downwards closed set with respect to involvement.

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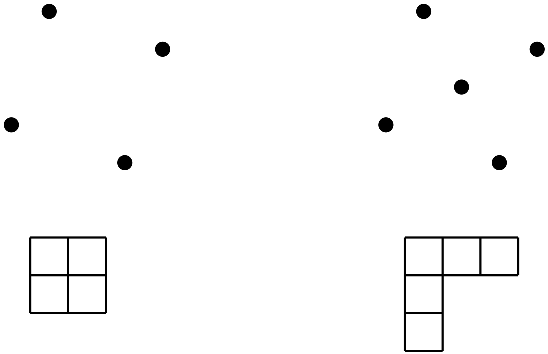
Question

Which permutation classes are closed under shape equivalence?

Ron M. Adin & Yuval Roichman, *Shape Avoiding Permutations*, JCTA 97 (2002), 162-176.

An annoying example



Domination and Greene's Theorem

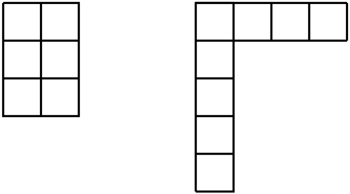
- A Young diagram λ *dominates* a Young diagram τ if, for each k, λ has at least as many cells in the first k rows as τ does.
- A Young diagram λ *doubly dominates* a Young diagram τ if, for each k, λ has at least as many cells in the first k rows as τ does, and the same is true for columns.

Theorem

Let π be a permutation. The sum of the lengths of the first k rows (columns) of $\mathrm{sh}(\pi)$ is equal to the length of the longest subpermutation of π that can be written as a union of k increasing (decreasing) subsequences.

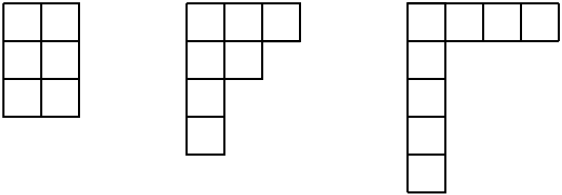
Curtis Greene, *An extension of Schensted's theorem*, Adv. Math. 14 (1974), 254-265.

But



- The second shape doubly dominates the first.
- No permutation of the first shape can be involved in one of the second.

Fortunately



- Form a doubly dominated chain
- There is a permutation of the first shape involved in one of the second
- And of the second involved in the third

Coverings in double domination

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- Any others?

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- Any others?
- No

New shapes from old

- Young diagrams can be concatenated (and rejustified) horizontally or vertically by simple operations on representative permutations
- Using this, any cover relation in the double domination order for Young diagrams can be witnessed by an involvement, provided that the single non-trivial cover from the 2×2 square can be.
- The annoying example is of some use after all!

So

Theorem

A permutation class is closed under shape equivalence if and only if it consists of all the permutations whose shapes belong to some downwards closed set in the double domination relation for Young diagrams.

Question

What is the refinement of Greene's theorem that characterises exactly when, given two Young diagrams, there is a permutation of the first shape involved in one of the second?