New techniques in permutation class enumeration

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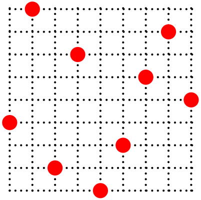
Permutation classes

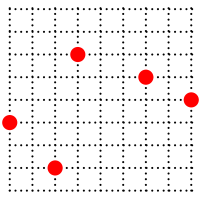
Definition

A permutation class is a collection of permutations, $\mathcal{C}$, with the property that, if $\pi \in \mathcal{C}$ and we erase some points from its plot, then the permutation defined by the remaining points is also in $\mathcal{C}$.



$492713685 \in \mathcal{C}$

implies



$21543 \in \mathcal{C}$

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- If X is a set of permutations, then $\mathrm{Av}(X)$ is the permutation class consisting of those permutations which do not dominate any permutation of X
- Many early results in the area were of the form “for a specific small set X of short permutations, the enumeration of $\mathrm{Av}(X)$ is …”
- We have tried to build a more general framework based on understanding the structure of (some) permutation classes, in which case enumerative results are a consequence, and not an end in themselves.

New techniques: simple permutations

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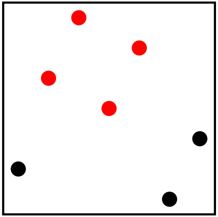

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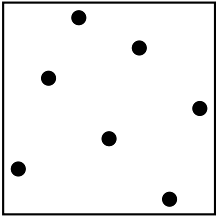

Simple

Finitely many simple permutations

Theorem

If a class has only finitely many simple permutations then it has an algebraic generating function.

- A and Atkinson (2005)
- Effective ‘in principle’, i.e. an algorithm for computing a defining system of equations for the generating function
- Some interesting corollaries, e.g. if a class has finitely many simples and does not contain arbitrarily long decreasing permutations then it has a rational generating function
- *“The prime reason for giving this example is to show that we are not necessarily stymied if the number of simple permutations is infinite.”*

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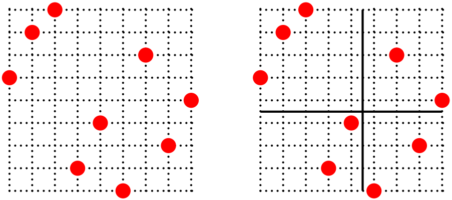
New techniques: grid classes

- The notion of *griddable* class was central to Vatter's characterization of small permutation classes
- Loosely, a griddable class is associated with a matrix whose entries are (simpler) permutation classes
- All permutations in the class can be chopped apart into sections that correspond to the matrix entries



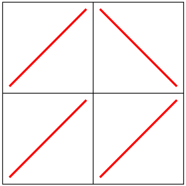
Geometric monotone grid classes

In a *geometric* grid class, the permutations need to be drawn from the points of a particular representation in $\mathbb{R}^2$

Theorem (A, At, Bouvel, Ruškuc and V (to appear TAMS))

Every geometrically griddable class:

- *is partially well ordered;*
- *is finitely based;*
- *is in bijection with a regular language and thus has a rational generating function.*



Beyond grid classes

Results from *Inflations of Geometric Grid Classes of Permutations*, A, R and V (arxiv.org/abs/1202.1833):

- Let $\langle \mathcal{C} \rangle$ denote the closure of $\mathcal{C}$ under inflation
- If $\mathcal{C}$ is geometrically griddable, then every subclass of $\langle \mathcal{C} \rangle$ is finitely based and partially well ordered
- If $\mathcal{C}$ is geometrically griddable, then every subclass of $\langle \mathcal{C} \rangle$ has an algebraic generating function
- Every small permutation class has a rational generating function

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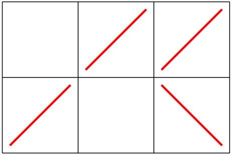
New tool: *PermLab*

- *PermLab* is a Java application that provides a workbench for exploration centred around permutation patterns.
- In casual use, a GUI supports various types of investigations.
- For studying conjectures, constructing examples, etc. the underlying class structures effectively provide an extensible domain specific language for more detailed investigations.
- `www.cs.otago.ac.nz/PermLab`

Case study: $\mathrm{Av}(4312, 3142)$

- Every simple permutation in this class lies in the geometric grid class:



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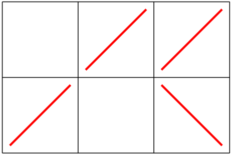

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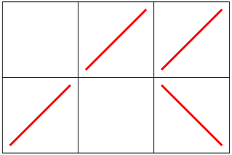

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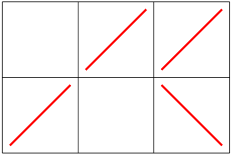

- This yields a regular language for the simple permutations
- The allowed inflations of these permutations are easily described, providing a recursive description of the class
- And in turn, a generating function:

$$\begin{aligned}(x^3 - 2x^2 + x)f^4 &+ (4x^3 - 9x^2 + 6x - 1)f^3 \\ &+ (6x^3 - 12x^2 + 7x - 1)f^2 \\ &+ (4x^3 - 5x^2 + x)f \\ &+ x^3 = 0\end{aligned}$$

Where to from here?

- Underlying the main results on geometric grid classes and their inflations is a notion of *natural encoding*, in this case of permutations by words, which may be applicable to other types of combinatorial structure particularly those carrying a linear order.
- Use of non-obvious symmetries between simple permutations to produce classes of *Wilf-equivalences*.
- Extensions to *infinite grid classes*.

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